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A SPECTRAL METHOD ON TETRAHEDRA USING RATIONAL BASIS FUNCTIONS

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Abstract. A spectral method using fully tensorial rational basis functions on tetrahedron, obtained from the polynomials on the reference cube through a collapsed coordinate transform, is proposed and analyzed. Theoretical and numerical results show that the rational approximation is as accurate as the polynomial approximation, but with a more effective implementation.

Key Words. Spectral methods on tetrahedra, rational basis functions, spectral accuracy.

1. Introduction

Spectral/*hp* element methods, which are capable of extending the merits of spectral methods to complex geometries, have become increasingly popular in computational fluid dynamics, atmospheric modeling and many other fields [6, 15, 5]. While the quadrilateral/hexahedral spectral element methods (QSEM) have achieved tremendous advances since the 80s [21, 18], considerable progress has been made recently in the triangular/tetrahedral element methods (TSEM). The TSEM have proven to be more flexible for complex domains and for adaptivity, and the currently existing approaches can be roughly classified as (i) the use of Koornwinder-Dubiner polynomials [7, 23, 15]; (ii) approximations by non-polynomials on triangular elements [3, 13], and (iii) approximations by polynomials on triangular elements using special nodal points such as Fekete points [14, 24, 19].

Although the use of polynomials on triangles/tetrahedra seems to be natural, this also brings the loss of some flexibility and some difficult implementation issues. For example, the Koornwinder-Dubiner polynomial basis functions, obtained from the collapsed transform, are based on a warped tensor product, which is more complicated in implementation and analysis than the standard tensorial case. However, if one drops the requirement of being polynomials on the triangular/tetrahedral elements, such issues can be circumvented. In a very recent paper [22], we proposed a fully tensorial TSEM using rational basis functions obtained from the polynomials in the reference square through a collapsed coordinate transform. This approach was shown to be at least as accurate as the warped tensorial TSEM using Koornwinder-Dubiner polynomials, and be able to be effectively implemented as

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the QSEM due to the fully tensorial nature and the availability of the nodal basis. In this paper, we discuss the generalization of this method to the case of three dimensional tetrahedron with an aim towards an adaptive element method on unstructured meshes. The extension to three dimensions is nontrivial for several reasons. Firstly, the collapsed transform from a tetrahedron to the reference cube induces severer singularities (i.e., two faces of the cube are collapsed into one edge and one vertex of the tetrahedron) than that of the two dimensional case. Hence, much care has to be taken for dealing with the singularities in both implementations and analysis. On the other hand, the complication of geometry leads to some additional difficulties for the construction of modal/nodal basis functions, and numerical implementations as well.

The outline of the paper is as follows. In Section 2, we introduce the collapsed coordinate transform and the rational basis functions. We also present some results on the approximation properties of the new basis in Sobolev spaces. In Section 3, we implement the rational spectral methods for some model equations on tetrahedron. The final section is for the extension to the tetrahedral spectral elements and some discussions. We end this section with some notations to be used throughout the paper.

- Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain, and ω be a generic positive weight function which is not necessary in $L^1(\Omega)$. Denote by $(u,v)_{\omega,\Omega} := \int_\Omega uv\omega d\Omega$ the inner product of $L^2_\omega(\Omega)$ whose norm is denoted by $\|\cdot\|_{\omega,\Omega}$. For any $m \geq 0$, we use $H^m_\omega(\Omega)$ and $H^m_{0,\omega}(\Omega)$ to denote the usual weighted Sobolev spaces, whose norm and semi-norms are denoted by $\|u\|_{m,\omega,\Omega}$ and $|u|_{m,\omega,\Omega}$, respectively. In case of no confusion would arise, ω (if $\omega \equiv 1$) may be dropped from the notations.
- Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{Z}^-$ the set of negative integers. For any $N \in \mathbb{N}$, we set $I = (-1,1)$ and denote by $\mathcal{P}_N(I)$ the set of all polynomials of degree $\leq N$, and set $\mathcal{P}^0_N(I) := \big\{\phi \in \mathcal{P}_N(I) : \phi(\pm 1) = 0\big\}$.
- We use the expression $A \lesssim B$ to mean that $A \leq cB$, where c is a generic positive constant independent of any function and of any discretization parameters.

2. Rational basis functions and approximations on tetrahedra

We introduce in this section a family of orthogonal rational basis functions on tetrahedra, and study its approximation properties in Sobolev spaces.

2.1. The collapsed coordinate transform. It is known that there exists an affine mapping between the reference tetrahedron:

$$\mathcal{T} = \big\{(x,y,z) : \ 0 < x, y, z, x+y+z < 1\big\}, \tag{2.1}$$

and any arbitrary tetrahedron $\mathcal{T}_P$ with vertices $P_0 = (u_0, v_0, w_0)^{\mathsf{tr}}$, $P_1 = (u_1, v_1, w_1)^{\mathsf{tr}}$, $P_2 = (u_2, v_2, w_2)^{\mathsf{tr}}$ and $P_3 = (u_3, v_3, w_3)^{\mathsf{tr}}$, which takes the form

$$\begin{cases} u = u_0(1-x-y-z) + u_1 x + u_2 y + u_3 z, \\ v = v_0(1-x-y-z) + v_1 x + v_2 y + v_3 z, \\ w = w_0(1-x-y-z) + w_1 x + w_2 y + w_3 z. \end{cases}$$

In view of this, we shall restrict our attentions to the reference tetrahedron $\mathcal{T}$. We also introduce a second coordinate $(\xi, \eta, \zeta)-$system on the reference cube $\mathcal{Q} :=$