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NOVEL FINITE DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF TWO-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In the present article, a new methodology has been developed to solve two-dimensional (2D) Navier-Stokes equations (NSEs) in new form proposed by Pukhnachev (J. Appl. Mech. Tech. Phys., 45:2 (2004), 167–171) who introduces a new unknown function that is related to the pressure and the stream function. The important distinguish of this formulation from vorticity-stream function form of NSEs is that stream function satisfies to the transport equation and the new unknown function satisfies to the elliptic equation. The scheme and algorithm treat the equations as a coupled system which allows one to satisfy two conditions for stream function with no condition on the new function. The numerical algorithm is applied to the lid-driven cavity flow as the benchmark problem. The characteristics of this flow are adequately represented by the new numerical model.

Key Words. Navier-Stokes equations, incompressible viscous flow, finite-difference scheme.

1. Introduction

There are many numerical schemes for the solution of the Navier-Stokes Equations (NSEs) representing incompressible viscous flows. Some of these are schemes utilize primitive variables (velocity-pressure), vorticity-stream function, stream function (biharmonic equation), and vorticity-velocity formulation. The primary difficulty in obtaining numerical solutions with primitive variable formulation is that there is no evolution equation for the pressure variable. To avoid the troubles associated with primitive variable approach, stream function vorticity and vorticity-velocity formulations of the NSEs are widely used. Unfortunately, the correct boundary values of vorticity are not always easy to get. In the present study we use a new form of the NSEs. Aristov and Pukhnachev [1] and Pukhnachev [7] proposed new form of the NSEs for the case of axisymmetric and 2D flows, respectively. The 2D NSEs in the terms of new unknown functions contain one transport equation for the stream function and one elliptic equation for the new unknown function. This system only resembles the vorticity and stream function's form but the physical meaning of the coupling function is different from the vorticity. We have constructed finite-difference scheme for the NSEs in the new form. Our algorithm treats the equations as a coupled system which allows us to satisfy two conditions for stream function with no condition on the new unknown function. The proposed

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scheme can easily be extended to solving axisymmetric NSEs. The performance of the proposed method is investigated by considering well-known benchmark problem. A test case involves simulating a 2D lid-driven cavity flow at Reynolds number $Re \leqslant 1000$, when the motion is strictly laminar and steady. Several numerical investigations have been reported the characteristics of this cavity flow. Moreover, viscous fluid flow inside a driven cavity has been a common experiment approach used to check or improve numerical techniques (see for example, [2, 3, 4, 5, 6, 8]).

The content of this paper is organized as follows. In the next section, we derive new formulation of the NSEs with no-slip boundary conditions. Section 3 briefly describes the problem used for the test case and detailed description of numerical algorithm. The results of validation of the finite-difference scheme are presented in Section 4, where we make a detailed comparison with available numerical and experimental data.

2. The New Formulation of Navier-Stokes equations

To make paper self completed we first represent the transformation of viscous incompressible NSEs in 2D to a new form. The viscous incompressible flow is governed by the NSEs in a Cartesian coordinate system (x, y),

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x + \nu\big(u_{xx} + u_{yy}\big), \tag{1}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y + \nu\big(v_{xx} + v_{yy}\big), \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where u and v are the velocity components in $x-$ and $y-$ directions, respectively; p is the pressure, ρ is the fluid density, and ν is the kinematic viscosity. The fluid is subjected to potential external forces. In 2D, the constrain of incompressibility $\nabla \cdot \mathbf{v} = 0$ can be satisfied exactly by expressing velocity vector in terms of stream function ψ according to

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4}$$

New form of NSEs is based on the following observation. The substitution of Eq. (4) into Eq. (1) yields

$$\frac{\partial}{\partial y}\left(\psi_t - \psi_x\psi_y - \nu\Delta\psi\right) + \frac{\partial}{\partial x}\left(\frac{1}{\rho}p + \psi_y^2\right) = 0, \tag{5}$$

where

$$\Delta \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Therefore, there is a function Φ satisfies the relations

$$\frac{1}{\rho}p = -\psi_y^2 + \Phi_y, \tag{6}$$

and

$$\psi_t - \psi_x\psi_y + \Phi_x = \nu\Delta\psi. \tag{7}$$

Differentiating Eq. (6) and Eq. (7) with respect to y and x, respectively, and substituting the resulting expressions into Eq. (2), where u and v are expressed in terms of ψ, we obtain

$$\Delta\Phi = 2\psi_y\Delta\psi. \tag{8}$$