INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 1, Pages 87–107

DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD WITH INTERIOR PENALTIES FOR CONVECTION DIFFUSION OPTIMAL CONTROL PROBLEM

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(Communicated by Wenbin Liu)

Abstract. In this paper, a discontinuous Galerkin finite element method with interior penalties for convection-diffusion optimal control problem is studied. A semi-discrete time DG scheme for this problem is presented. We analyze the stability of this scheme, and derive a priori and a posteriori error estimates for both the state and the control approximation.

Key Words. discontinuous Galerkin method, convection-diffusion equation, optimal control problem, a priori error estimates, a posteriori error estimates

1. Introduction

Finite element approximation of optimal control problems has been an important topic in engineering design work. There has been extensive theoretical and numerical studies for standard finite element approximation of various optimal control problems. For instance, for the optimal control problems governed by some linear elliptic or parabolic state equations, a priori error estimates of the finite element approximation were established long ago, see [1, 2, 3, 4, 5]. Furthermore, a priori error estimates were established for the finite element approximation of some important flow control problems in [6]. Some recent progress in a priori error estimates can be found in [7, 8] and in [9, 10, 11, 12], for a posteriori error estimates. Systematic introduction of the finite element method for PDEs and optimal control problems can be found in, for example, [13], [14] and [15].

In recent years, the discontinuous Galerkin methods have been proved very useful in solving a large range of computational fluid problems ([16, 17, 18]). They are preferred over standard continuous Galerkin methods because of their flexibility in approximating globally rough solutions, their local mass conservation, their possible definition on unstructured meshes, their potential for error control and mesh adaptation.

The idea of using penalty terms in a finite element method is not new. Baker [19] was the first one who used interior penalty with nonconforming elements for elliptic equations. Douglas and Dupont [20] analyzed a method which used interior penalties on the derivatives with conforming elements for linear elliptic and parabolic problems. Inspired by [19], Wheeler [21] presented an interior penalty method for second order linear elliptic equations. Closest to [21], Arnold [22] formulated a semi-discrete discontinuous Galerkin method with interior penalty for second order nonlinear parabolic equations.

Received by the editors September 9, 2008 and, in revised form, July 5, 2009.
2000 *Mathematics Subject Classification.* 65N30, 49J20 .
This research was supported by the NSF of China(No.10571108) and SRF for ROCS, SEM .

These methods [20, 21, 22] generalized a method by Nitsche [23] for treating Dirichlet boundary condition by the introduction of penalty terms on the boundary of the domain. Applications of these methods to flow in porous media were presented by Douglas, Wheeler, Darlow and Kendall in [24]. These methods frequently referred to as interior penalty Galerkin schemes.

In general, penalty terms are weighted L^2 inner products of the jumps in the function values across element edges. The primary motivation of including interior penalties is to impose approximate continuity. These terms enable closer approximation of solutions which varies in character from one element to another and allow the incorporation of partial knowledge of the solution into the scheme. Numerical experiments have clearly demonstrated the value of penalties for solving certain problems (see, e.g., [20]). New applications of discontinuous Galerkin method with interior penalties to nonlinear parabolic equations were introduced and analyzed by Rivière and Wheeler ([17, 25, 26]). It was shown that the method in ([17, 25, 26]) was elementwise conservative, and a priori and a posteriori error estimates in higher dimensions were derived.

Optimal control for convection-diffusion equation is widely met in practical applications. For example, in Environmental Sciences, some phenomena modelled by linear convection-diffusion partial differential equations are often studied to investigate the distribution forecast of pollutants in water or in atmosphere. In this context it might be of interest to regulate the source term of the convection-diffusion equation so that the solution is as near as possible to a desired one, e.g., to operate the emission rates of industrial plants to keep the concentration of pollutants near (or below) a desired level. This problem can be conveniently accommodated in the optimal control framework for convection-diffusion equation. Some existing works ([27, 28, 29, 30]) focus on the stationary convection dominated optimal control problem. They used several stabilization methods to improve the approximation properties of the pure Galerkin discretization and to reduce the oscillatory behavior, e.g SUPG method in [27], stabilization on the Lagrangian functional method in [28], reduced basis (RB) technique in [29]. However to our best knowledge, there has been a lack of proper study for general time-dependent convection-diffusion optimal control problem.

The purpose of this paper is to extend the discontinuous Galerkin method with interior penalties in [17, 22] to time-dependent convection-diffusion optimal control problem. A semi-discrete time DG scheme for this problem is presented. The first difficulty for our problem is to derive the discretization of the co-state equation and the optimality conditions. We first establish the semi-discrete time DG scheme for the state equation, prove the stability and the existence of this scheme, then apply the theory of optimal control problem (see, [31]) to this scheme for deriving the discretization of the co-state equation and the optimality conditions. The DG scheme of state equation is complicated so that it is much more difficult to derive the discretized co-state equation, which is quite complicated. The complexity of the DG schemes of the state and the co-state equation also leads to the difficulties in deriving a priori error estimates and a posteriori error estimates later. To our knowledge, this paper appears to be the first trial to approximate convection-diffusion optimal control problem by using the Discontinuous Galerkin method with interior penalties.

The outline of the paper is as follows. In Section 2, we first briefly introduce convection-diffusion optimal control problem and optimality conditions. In Section 3, we give some definitions, then use discontinuous Galerkin method with interior penalties to construct a semi-discrete approximate scheme for convection-diffusion optimal control problem.