INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 4, Pages 711–723

SUPERCONVERGENCE BY L^2-PROJECTIONS FOR STABILIZED FINITE ELEMENT METHODS FOR THE STOKES EQUATIONS

JIAN LI, JUNPING WANG, AND XIU YE

Abstract. A general superconvergence result is established for the stabilized finite element approximations for the stationary Stokes equations. The superconvergence is obtained by applying the L^2 projection method for the finite element approximations and/or their close relatives. For the standard Galerkin method, existing results show that superconvergence is possible by projecting directly the finite element approximations onto properly defined finite element spaces associated with a mesh with different scales. But for the stabilized finite element method, the authors had to apply the L^2 projection on a trivially modified version of the finite element solution. This papers shows how the modification should be made and why the L^2 projection on the modified solution has superconvergence. Although the method is demonstrated for one class of stabilized finite element methods, it can certainly be extended to other type of stabilized schemes without any difficulty. Like other results in the family of L^2 projection methods, the superconvergence presented in this paper is based on some regularity assumption for the Stokes problem and is valid for general stabilized finite element method with regular but non-uniform partitions.

Key words. Stokes equations, Stabilized finite element method, Superconvergence, L^2 projection, least-squares method

1. Introduction

In the analysis and practice of employing finite element methods in solving the Navier-Stokes equations, the inf-sup condition has played an important role because it ensures a stability and accuracy of the underlying numerical schemes. A pair of finite element spaces that are used to approximate the velocity and the pressure unknowns are said to be stable if they satisfy the inf-sup condition. Intuitively speaking, the inf-sup condition is something that enforces a certain correlation between two finite element spaces so that they both have the required properties when employed for the Navier-Stokes or Stokes equations. It is well known that the two simplest elements P_1/P_0 (i.e., linear/constant) on triangle and Q_1/P_0 (i.e., bilinear/constant) on quadrilateral are not stable, and therefore can not be trusted when employed in practical computation. In contrast, many known stable elements do not look natural because their construction involves artificial or non-standard functions which are not commonly used/implemented in popular engineering code packages. To eliminate the inf-sup condition so that simpler and more natural finite element spaces can be used, stabilized finite element methods have been developed for the Stokes equations in the last two decades [14, 4, 15, 9, 16]. These methods are gaining more and more popularity in computational fluid dynamics.

The goal of this paper is to explore ways that improve the accuracy of the approximate solutions resulted from the stabilized finite element formulations for the Stokes equations. In particular, we are curious about postprocessing techniques that lead to new approximations with superconvergence. In the literature, there are number of techniques in the content of superconvergence [8, 10, 27, 24, 22, 18,

Received by the editors January 1, 2009 and, in revised form, March 22, 2009.
2000 *Mathematics Subject Classification.* 65N15,65N30, 76D07.
The third author of this research was supported in part by National Science Foundation Grant DMS-0813571.

31, 32, 30]. The main idea behind them is "cancelation", which, at least in all the existing results, are possible only for certain model problems with some strong, and perhaps impractical assumptions on the geometry of the finite element partitions, see for example [20, 21]. One exception in techniques of superconvergence is the L^2-projection method proposed and analyzed by Wang [28] for the standard Galerkin method. The relaxation on the mesh uniformity is the key difference between the L^2-projection method and all other methods in superconvergence. The L^2-projection method has been extended by Wang and Ye [29] to the Stokes equations, but only for finite element methods based on stable pairs. This paper aims at a study of superconvergence by using the L^2-projection method for the stabilized finite element method.

Briefly speaking, the result to be presented in this paper shows that it is possible to obtain numerical solutions with superconvergence for the stabilized finite element methods. However, there are essential differences between the stabilized finite element method and the standard Galerkin method. For example, one can obtain superconvergence for the L^2 projection of the pressure approximation, but not for the velocity approximation as one would get in the standard Galerkin method. However, the L^2-projection of a modified or corrected form of the velocity approximation is of superconvergent to the exact velocity. Our analysis shows that the correction comes from a scaled version of the residual which is exactly the stability term added to the Galerkin formula. If similar stability terms were added to the mass conservation equation, one would need to modify the pressure approximation as well in order to obtain superconvergence by using the L^2-projection method. The main contribution of the paper is that it provides a systematic approach for obtaining superconvergence when non-standard Galerkin methods are used.

The paper is organized as follows. In Section 2, we review a stabilized finite element formulation for the Stokes equations. In Section 3, we describe the general idea of the L^2 projection method in superconvergence. In Sections 4 and 5, we establish two super-approximation properties: one for the pressure and the other for the velocity unknown by using L^2-projections. Finally, in Section 6, we derive some new superconvergent results for the Stokes equations when approximated by using stabilized finite element methods.

2. Preliminaries and the stabilized finite element method

For simplicity, we consider the homogeneous Dirichlet boundary value problem for the Stokes equations. This model problem seeks unknown functions $\mathbf{u} \in H^1(\Omega)^d$ and $p \in L^2(\Omega)$ satisfying

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is an open bounded domain in the Euclidean space $\mathbf{R}^d (d = 2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$; $\mathbf{f}$ is a given function in $H^{-1}(\Omega)^d$; Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators respectively; $\nu > 0$ is a given constant representing the viscosity of the fluid. The given function/distribution $\mathbf{f} = \mathbf{f}(x)$ is the unit external volumetric force acting on the fluid at $x \in \Omega$. Without loss of generality, we assume that $\nu = 1$, $d = 2$, and Ω is polygonal in the rest of the paper.

The above description of the Stokes problem has assumed the standard notation for the Sobolev spaces $H^s(\Omega)$ which is the collection of distributions whose weak