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SUPERCONVERGENCE OF GALERKIN SOLUTIONS FOR HAMMERSTEIN EQUATIONS

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Abstract. In the present paper, we discuss the superconvergence of the interpolated Galerkin solutions for Hammerstein equations. With the interpolation post-processing for the Galerkin approximation x_h, we get a higher order approximation $I_{2h}^{2r-1}x_h$, whose convergence order is the same as that of the iterated Galerkin solution. Such an interpolation post-processing method is much simpler than the iterated method especially for the weak singular kernel case. Some numerical experiments are carried out to demonstrate the effectiveness of the interpolation post-processing method.

Key words. superconvergence, interpolation post-processing, iterated Galerkin method, Hammerstein equations, smooth and weakly singular kernels.

1. Introduction

In this paper, we investigate the superconvergence of the interpolated Galerkin solutions for Hammerstein equations with smooth and weakly singular kernels. As for Hammerstein equations, various numerical methods have been used to get the approximations. A variation of Nyström's method was proposed by Lardy [18]. Two different discrete collocation methods were proposed by Kumar [17] and Atkinson and Flores [3]. Brunner [7] discussed the connection between implicitly linear collocation methods and iterated spline collocation methods for Hammerstein equations, and then extended the results to a class of nonlinear Volterra-Fredholm integral equations. A degenerated kernel method for Hammerstein equations was introduced by Kaneko and Xu [14]. Kaneko, Noren, and Xu [13] used the product integration method and the collocation method to solve Hammerstein equations with weakly singular kernels, and got some superconvergence properties. A survey paper by Atkinson [2] gave more information about numerical solutions of Hammerstein equations. The superconvergence of the iterated Galerkin solutions for Hammerstein equations with smooth as well as weakly singular kernels was probed by Kaneko and Xu [16]. Moreover, the superconvergence of the iterated collocation method for Hammerstein equations with smooth as well as weakly singular kernels was studied by Kaneko, Noren, and Padila [11].

For Hammerstein equations, generally, the iterated post-processing method (see, for example, [4, 7, 11, 16]) is used to accelerate the approximation. If the kernel is sufficiently smooth, it is very easy to get the iterated Galerkin solutions. But if the kernel is weakly singular, there are many difficulties to

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get the iterated Galerkin solutions since the classical numerical quadrature is no longer valid.

In this paper, we use another type of acceleration method, the interpolation post-processing method, to get the same superconvergence. Applying the interpolation post-processing to the Galerkin approximation x_h, we get a higher accuracy approximation $I_{2h}^{2r-1}x_h$ (which is named the interpolated Galerkin solution throughout this paper), whose convergence order is the same as that of the iterated Galerkin solution. Furthermore, the interpolation post-processing method is simpler than the iterated post-processing method since we just need to interpolate x_h at some nearby points to get the interpolated Galerkin solution instead of computing a nonlinear integral for each subinterval which is especially difficult for the weakly singular kernel.

The interpolation post-processing technique can be used to improve the approximate rate of finite element solutions for various partial differential equations, integral equations, and integro-differential equations, and the corresponding work has been contained in some papers (such as [22, 26]) and some monographs (see [20, 21] for example). It has been found that this technique is both simple and of higher accuracy. For Hammerstein equations, Huang and Zhang [10] applied the interpolation post-processing to collocation solutions and obtained the same superconvergence as that of the iterated collocation method.

Here is the outline of the remaining sections. The Galerkin method and the iterated Galerkin method for Hammerstein equations are presented in Section 2. And some materials for the approximation theory are also reviewed in this section to make the paper self-contained. In Section 3, main results about the superconvergence of interpolated Galerkin solutions, instead of the iterated collocation solutions, are obtained. Finally, numerical experiments are listed in Section 4 to show the efficiency of the interpolation post-processing method.

2. The Iterated Galerkin Method

In this section, the Galerkin method and the iterated Galerkin method are considered for the following Hammerstein equation

$$x(t) - \int_0^1 k(t,s)\psi(s,x(s))ds = f(t), \qquad 0 \le t \le 1, \tag{2.1}$$

where k, f and ψ are known functions and x is the function to be determined. Define $k_t(s) \equiv k(t,s)$ for $t,s \in [0,1]$ to be the t section of k. We assume throughout this paper unless stated otherwise, the following conditions on k, f, and ψ hold:

1. $\lim\limits_{t\to\tau} \|k_t - k_\tau\|_\infty = 0, \qquad \tau \in [0,1]$;
2. $M \equiv \sup\limits_{0\le t\le 1} \int_0^1 |k(t,s)|ds < \infty$;
3. $f \in C[0,1]$;