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LAPLACE TRANSFORMATION METHOD FOR THE BLACK-SCHOLES EQUATION

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Abstract. In this paper we apply the innovative Laplace transformation method introduced by Sheen, Sloan, and Thomée (IMA J. Numer. Anal., 2003) to solve the Black-Scholes equation. The algorithm is of arbitrary high convergence rate and naturally parallelizable. It is shown that the method is very efficient for calculating various option prices. Existence and uniqueness properties of the Laplace transformed Black-Scholes equation are analyzed. Also a transparent boundary condition associated with the Laplace transformation method is proposed. Several numerical results for various options under various situations confirm the efficiency, convergence and parallelization property of the proposed scheme.

Key Words. Black-Scholes equation, basket option, Laplace inversion, parallel method, transparent boundary condition

1. Introduction

As stock markets have become more sophisticated, so have their products. The simple buy/sell trades of the early markets have been replaced by more complex financial options and derivatives. These contracts can give investors various opportunities to tailor their dealings to their investment needs.

One of the main concerns about financial options is what the exact values of options are. For the simplest model in the case of constant coefficients, an exact pricing formula was derived by Black and Scholes, known as the Black-Scholes formula. However, in the general case of time and space dependent coefficients the exact pricing formula are not yet established, and thus numerical solutions have been used.

In order to describe an option price, let x, K, t and T denote the underlying asset price, the strike price, the time to maturity, and the expiry date of an option, respectively. As usual, σ and r represent the volatility of the underlying asset and the risk-free interest rate of the market, respectively. In this paper, we assume that σ and r depend on x only. Then a European option price $u(x,t)$ satisfies the Black-Scholes equation:

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru = 0, \quad (x,t) \in (0,\infty)\times(0,T], \tag{1.1}$$

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where an initial condition $u_0(x) = u(x,0)$ is given by the initial contract of an option. The basket option based on n assets $\mathbf{x} = (x_1, \ldots, x_n)$ satisfies

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} r x_i \frac{\partial u}{\partial x_i} + ru = 0, \quad (\mathbf{x}, t) \in (0,\infty)^n \times (0,T], \tag{1.2}$$

where $a_{ij} = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk}$, with σ_{ij} representing the corelation between the assets x_i and x_j.

Several numerical methods have been used for solving the Black-Scholes equation, for example in [32, 28] and [10] and the references therein, one can find popular numerical schemes for option pricing. Usually the time marching methods such as forward Euler, backward Euler and Crank-Nicolson schemes are used with a suitable spatial discretization scheme. In spite of the popularity of these time marching methods, a critical drawback of these schemes is that they usually require as many time steps as spatial meshes to balance the errors arising from discretization. In particular, for the estimation of basket options of reasonable size, the usual time marching schemes seem to be too slow in practice since the cost of solving an elliptic system to advance to a next time step is usually expensive. It is thus highly desirable to solve as small a number of elliptic solution steps as possible as well as to apply a very fast elliptic solver.

In this paper, we will focus on minimizing the number of elliptic solution steps by proposing the Laplace transformation method for the Black-Scholes equation, which is also naturally parallelizable. It will be shown that our method can dramatically reduce the computing time compared to the time marching schemes. Suitable contours should be chosen in order to have very fast convergence, and for this, we will estimate the resolvent of the Black-Scholes equation. Also, an exact transparent boundary condition will be given at which the infinite spatial domain is truncated.

There have been some related works in which the Laplace transformation method has been used, for instance in [7, 18, 25]. However, in these earlier papers the Laplace transformation method has been used to obtain the analytic solution of various options rather than to develop an efficient numerical scheme. In particular, in [18] the partial Laplace transformation is applied for American option pricing, and in [6, 24] the Mellin transformation which is similar to the Laplace transformation is used to evaluate the analytic solution of an option. Related with Laplace transformation methods there are other approaches based on the so-called $\mathcal{H}$-matrix approach; for instance, see [8, 9], and so on. Also, high-dimensional parabolic problems can be solved using sparse grids [11, 15, 16, 27]. Application of our Laplace transformation method using sparse grids to option pricing will also be interesting. Other approaches in the fast time-stepping methods can be found in [34, 20, 19].

In the following section, we will briefly describe the Laplace transformation method proposed by Sheen, Sloan, and Thomée in [30] with its numerical procedure and convergence. Then in §3 we will examine the properties of the Laplace transformed Black-Scholes equation including the solvability of the transformed equation, transparent boundary condition and the resolvent. Finally in §4 we will present several numerical results for various options and various situations with the parallelization property of the proposed scheme.