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FRONT TRACKING ALGORITHM FOR THE LIGHTHILL-WHITHAM-RICHARDS TRAFFIC FLOW MODEL WITH A PIECEWISE QUADRATIC, CONTINUOUS, NON-SMOOTH, AND NON-CONCAVE FUNDAMENTAL DIAGRAM

WENQIN CHEN, S.C. WONG, CHI-WANG SHU, AND PENG ZHANG

Abstract. We use a front tracking algorithm to explicitly construct entropy solutions for the Lighthill-Whitham-Richards traffic flow model with a flow-density relationship that is piecewise quadratic, continuous, non-smooth, and non-concave. The solution is exact if the initial condition is piecewise linear and the boundary conditions are piecewise constant. The algorithm serves as a fast and accurate solution tool for the prediction of spatio-temporal traffic conditions and as a diagnostic tool for testing the performance of numerical schemes. Numerical examples are used to illustrate the effectiveness and efficiency of the proposed method relative to numerical solutions that are obtained using a fifth-order weighted essentially non-oscillatory scheme.

Key words. LWR model, traffic flow, piecewise quadratic fundamental diagram, front tracking algorithm, WENO scheme.

1. INTRODUCTION

Lighthill and Whitham [23] provided one of the first published theories of the macroscopic modeling of highway traffic flow. Their theory was based on two relationships: a continuity equation and the fundamental relationship between the flow and density of a traffic stream. The continuity equation can easily be derived by considering the conservation of vehicles between any two locations on a road, which is why it is often called a conservation equation. As an assumed speed-density relationship is needed to solve and apply the continuity equation for traffic flow, studies on the relationships between the fundamental traffic stream variables are vital, and have been provided throughout the history of traffic flow study. With the continuity equation, a speed-density relationship, and the initial and boundary conditions of the traffic stream, the density at any location along a road can be determined. Richards [30] independently proposed the same continuum approach, albeit in a slightly different form. The key difference is that Richards focused on the derivation of shock waves with respect to density, whereas Lighthill and Whitham considered the same from the perspective of disruptions to traffic flow. Another difference between the two methods is that Richards adopted a linear speed-density relationship, whereas Lighthill and Whitham used a more general speed-density relationship. Because of the nearly simultaneous and independent development of the theory, the model has become known in the literature as the LWR model.

The LWR model, as a scalar hyperbolic conservation law, can be solved by approximating the fundamental diagram (or flux function) as a piecewise linear function [3], in which the solution of the Riemann problem is a step function (piecewise

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constants). If the initial data falls within the class of step functions, then an analytical solution of the Cauchy problem can be constructed by the superposition of simple Riemann problems. The solution will always remain piecewise constant, and will therefore always be within the same class of functions, because all wave interactions lead to new Riemann problems. By solving the Riemann problems that arise each time two or more waves interact, a global solution can be established. In the general case, approximating the flux function by a sequence of piecewise linear functions and the initial data by a sequence of step functions gives a compact sequence of approximate solutions that converge to the solution of the Cauchy problem. Similarly, Newell [27–29] assumed a triangular or trapezoidal shape of the fundamental diagram and proposed a simple graphical procedure to derive the analytical solution to the LWR model on incident detection using the concept of cumulative flow. These studies unintentionally share the same rationale as Dafermos' method. More recently, and inspired by Dafermos, Henn [6, 7] proposed a solution algorithm for the LWR model known as the wave tracking scheme that is based both on the piecewise linear approximation of the fundamental flow-density relationship and on an explicit tracking of waves, and further implemented the scheme to evaluate the impact of incidents on the road.

Lucier [26] extended Dafermos' method to approximate the flux function using a parabolic spline approximation, in which the piecewise quadratic functions are continuously differentiable with discontinuous second derivatives at the breakpoints. Holden et al. [8, 9] enhanced Dafermos' method and showed that even in infinite time, there are only a finite number of constant states. They also proved that the construction is well defined for non-convex flux functions. They called the method front tracking – front referring to the discontinuities and tracking to the process of computing collisions and resolving interactions. Front tracking has proved to be a very robust numerical method for scalar, one-dimensional conservation laws. Kunick [21] proved an explicit representation formula for the solution of a one-dimensional hyperbolic conservation law with a non-convex flux function but monotone initial data based on the polygonal method of Dafermos. Other developments and applications of the front tracking method can also be found in [1, 11, 14–17].

Unaware of the earlier development of the front tracking method [10], Wong and Wong [35] rediscovered the method of Lucier [26] for solving a scalar hyperbolic conservation law, and determined the formation and propagation of shocks on a homogeneous highway subject to general boundary conditions assuming a linear speed-density relationship (or parabolic flux function). The method of Wong and Wong [35] can therefore be considered to be a special case of the method of Lucier [26] in which the solution is exact if the fundamental diagram is a parabolic flux function, the initial condition is piecewise linear, and the boundary conditions are piecewise constant. In both Wong and Wong [35] and Lucier [26], explicit expressions that describe the relationship between the density, space, and time of different scenarios (characteristics, fan, and shock) were derived to allow the evolution of traffic density in space and time to be precisely determined, although Lucier's focus was on the theoretical proof of the convergence rate of the algorithm when a general flux function is approximated by a number of piecewise quadratic functions. More recently, Lu et al. [25] proposed an improved front tracking algorithm that adopts a piecewise quadratic, continuous, and concave fundamental diagram. As their algorithm does not require the piecewise quadratic function to be continuously differentiable at the junction points, they improved on the method proposed