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FINITE ELEMENT APPROXIMATION OF THE GRADIENT FLOW FOR A CLASS OF LINEAR GROWTH ENERGIES WITH APPLICATIONS TO COLOR IMAGE DENOISING

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This paper is dedicated to Professor Roland Glowinski on the occasion of his seventieth birthday

Abstract. This paper concerns with the finite element approximation of a nonlinear second order parabolic system which describes the L^2-gradient flow for a class of linear growth energy functionals. Besides their appeals in differential geometry and calculus of variations, linear growth energy functionals and their gradient flows also arise naturally from emerging applications of image processing such as color image denoising. In this paper, we introduce a family of variational models for color image denoising which minimize linear growth energy functionals of maps into the unit sphere in $\mathbf{R}^3$. These models generalize the popular 1-harmonic map model which has been studied intensively in recent years. To compute the solutions of the variational models, we first derive their L^2-gradient flow equations and then introduce some fully discrete implicit finite element method for the gradient flow equations. It is proved that the proposed finite element method is uniquely solvable and absolutely stable, and the finite element solution converges to the PDE solution as the mesh sizes tend to zero. Numerical experiments are presented to demonstrate the effectiveness of the proposed variational models for color image denoising and to show the efficiency of the proposed finite element method. A numerical comparison of the proposed models with the channel-by-channel model is also presented.

Key Words. Linear growth energy functionals, gradient flow, p-harmonic maps, BV functions, color image denoising, finite element methods

1. Introduction

Image denoising and restoration are two most basic tasks in low level image processing. Tremendous progresses have been made in this area, particularly for gray images, in the past two decades. In addition to the further development in the traditional methods, techniques and algorithms, there has been an explosive development and growth of image processors based on partial differential equations (PDEs) and variational approach (cf. [4, 23, 6, 21] and the references therein). Compared with the traditional approaches, PDE and variational methods have remarkable advantages in both theory and computation. It allows to directly handle and process visually important geometric features such as gradients, tangents and curvatures, and to model visually meaningful dynamic process such as linear

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and nonlinear diffusion. Computationally, it can greatly benefit from the existing wealthy amount numerical methods for PDEs (cf. [4, 20, 19, 21]).

A color image is often represented by a vector-valued image function $\mathbf{I}(x) = (r(x), g(x), b(x))$, where $r(x), g(x)$, and $b(x)$ denote the intensity values of three primary colors (Red, Green, Blue (RGB)) at an image pixel x. Suppose that $\mathbf{I}(x)$ describes a given scene which contains some random noise, the goal of color image denoising is to remove the noise such that the recovered image is as close as possible to the true image. One of the classical approaches for color image denoising is the median filter which works especially well for enhancing edges [18]. Recently, the median filter has been generalized to denoising chromaticity features on the unit sphere [28, 29]. Another approach is to treat the RGB color system directly as a vector space and to denoise it channel-by-channel [22, 8]. Most recent studies have proposed the use of the chromaticity and brightness decomposition (CBD). Indeed, the CBD approach has received a lot attention lately since it is well suited for denoising, edge detection, and segmentation, see [11, 27, 26, 28] and the references therein.

Mathematically, the CBD is nothing but the polar decomposition of the vector-valued image function. That is, we write

$$\mathbf{I}(x) = \rho(x)\mathbf{g}, \tag{1}$$

where

$$\rho(x) := |\mathbf{I}(x)| = \sqrt{r(x)^2 + g(x)^2 + b(x)^2}, \qquad \mathbf{g} := \frac{\mathbf{I}(x)}{\rho(x)}. \tag{2}$$

Therefore, $\rho(x)$, called the *brightness* of the image, is the length of the RGB color vector, and $\mathbf{g}$, called the *chromaticity* of the image, denotes the direction of the color vector which must lie on the unit sphere $\mathbf{S}^2$. One advantage of CBD approach is that it allows one to denoise the chromaticity and the brightness separately by different methods. For instance, one can use the well-known total variation (TV) model of Rudin-Osher-Fatemi [20] (also see [15, 16]) to denoise the rightness $\rho(x)$ and use another method to denoise the chromaticity $\mathbf{g}$. A number of authors have addressed color image denoising using directional diffusion of $\mathbf{R}^n$ vectors [8, 22, 24]. All these works extended the well established scalar diffusion flows [3, 20] in different forms for the vector-valued image and do not separate the chromaticity and brightness. Blomgren and Chan [8] proposed a new definition of the total variation norm for vector-valued functions which is the extension of the scalar TV norm and applied this new TV norm to restore color images. Some authors have used p-harmonic map flows for chromaticity denoising [30, 26, 5]. Most of these works considered the case $1 < p < \infty$ [30, 26]. Barrett, Feng and Prohl [5] proved the existence of weak solutions for the whole spectrum $1 \leq p < \infty$. Feng [14] extended the 1-harmonic map results of [5] to gradient flows of linear growth functionals. Chan, Kang and Shen [10] applied the general framework of non-flat TV denoising model [11] to chromaticity denoising.

The primary goals of this paper are to introduce a family of variational models for color image denoising and to develop some fully discrete finite element method for computing solutions of the proposed models. The proposed models use convex linear growth functionals instead of the p-energy functional (cf. Section 2). We