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# AN OPTIMAL-ORDER ERROR ESTIMATE TO THE MODIFIED METHOD OF CHARACTERISTICS FOR A DEGENERATE CONVECTION-DIFFUSION EQUATION

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**Abstract.** We prove *a priori* error estimates in a weighted energy norm to the modified method of characteristics (MMOC) for time-dependent convection-diffusion equations with degenerate diffusion. The convergence rates are independent of the lower bound of the diffusion. In other words, these estimates hold uniformly with respect to the degenerate diffusion.

**Key Words.** Convergence analysis, degenerate convection-diffusion equations, Eulerian-Lagrangian methods, interpolation of spaces, optimal-order estimates

## 1. Introduction

Time-dependent advection-diffusion equations arise in mathematical models of porous medium flow and transport processes, including petroleum reservoir simulation, environmental modeling, and other applications. In such applications as immiscible displacement of oil by water in a secondary oil recovery process in petroleum industry or a groundwater transport process involving a non-aqueous phase liquid (NAPL), the corresponding governing equation is a degenerate time-dependent nonlinear advection-diffusion equation for the saturation of the invading phase. The diffusion in the saturation equation is due to capillary pressure effect, which could vanish or exhibit significant effect depending on whether the wetting phase or the nonwetting phase occupies the pore space [3, 5, 16, 19]. On the other hand, subsurface geological formations often consist of layered media, in which the diffusion parameters could vary by several orders of magnitude. In all of these applications, the governing equations could be convection-dominated in part of the domain while diffusion-dominated in the other part. Consequently, these problems admit solutions with moving fronts and complex structures and present serious mathematical and numerical difficulties.

Classical finite difference or finite element methods tend to generate numerical solutions with nonphysical oscillations, while classical upwind methods often produce numerical solutions with excessive numerical diffusion that smears out the fronts and generates spurious effects related to grid orientation [9, 14, 16]. Eulerian-Lagrangian methods provide an alternative approach for numerically solving time-dependent advection-diffusion equations. These methods combine the advection and capacity terms in the governing equations to carry out the temporal discretization in the Lagrangian coordinates, and discretize the diffusion term on a fixed mesh in the Eulerian coordinates [6, 11, 13, 24]. They symmetrize the governing equation and stabilize their numerical approximations. Moreover, they generate accurate numerical solutions and significantly reduce the numerical diffusion and

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grid-orientation effect present in upwind methods, even if large time steps and coarse spatial meshes are used. Eulerian-Lagrangian methods were shown to be very competitive in terms of accuracy and efficiency [11, 26, 28].

The modified method of characteristics was one of the pioneering methods in the class of Eulerian-Lagrangian methods and was proposed and analyzed in early 1980s [13]. Since then the MMOC has been successfully applied to the numerical simulation of coupled systems in miscible displacement and immiscible displacement in petroleum industry [9, 14, 17]. Subsequently, various improvements of the MMOC were developed, including the modified method of characteristics with adjusted advection (MMOCAA) [11, 12], the Eulerian-Lagrangian localized adjoint method (ELLAM) [1, 6, 24, 25, 28], the characteristic mixed finite element method (CMFEM) [2, 33], and the Eulerian-Lagrangian discontinuous Galerkin method (ELDG) [32] in the context of linear advection-diffusion equations, single-phase miscible displacement processes, immiscible two-phase flow, and multiphase multicomponent flow and transport processes in porous media.

Extensive research has been carried out on the convergence analysis and error estimates for the MMOC [13], the MMOCAA [12], the CMFEM [2], the ELLAM [22, 23, 27], and the ELDG [32]. However, the generic constants in these estimates depend inversely on the vanishing parameter $\varepsilon$ and so will blow up as $\varepsilon$ tends to zero. These estimates fail to reflect the uniformly optimal-order convergence rates of the Eulerian-Lagrangian methods observed numerically. An $\varepsilon$-uniform suboptimal-order error estimate was proved for the MMOC scheme for a time-dependent advection-diffusion equation with an incompressible velocity field $\mathbf{v}$ and a nonstandard boundary condition $\mathbf{v} = \mathbf{0}$ and the diffusion of the form $\varepsilon\Delta u$ [4]. Subsequently, $\varepsilon$-uniform estimates for the MMOC, the MMOCAA, the ELLAM, and the ELDG schemes for time-dependent advection-diffusion equations with the diffusion of the form $\varepsilon D(x,t)$ and with a periodic boundary condition or a general flux boundary condition [29, 30, 31] were obtained. However, all of these $\varepsilon$-uniform estimates depend heavily on the lower bound $D_{min}$ and upper bound $D_{max}$ of the diffusion coefficient $D(x,t)$, although they are $\varepsilon$-independent. A suboptimal error estimate was proved for the MMOC scheme for a degenerate time-dependent advection-diffusion equation [10].

In this paper we prove *a priori* optimal-order error estimates in a weighted energy norm to the MMOC scheme for degenerate time-dependent convection-diffusion equations with a degenerate diffusion. The convergence rates are independent of the lower bound of the diffusion, and they do not require the upper bound of the diffusion to tend to zero at the same rate as in the case of vanishing diffusion coefficient $\varepsilon$. The rest of this paper is organized as follows. In §2 we recall preliminary results on Sobolev spaces and interpolation of spaces. In §3 we revisit the MMOC scheme. In §4 we prove optimal-order error estimates in a weighted energy norm to the MMOC scheme. In §5 we prove auxiliary lemmas, which were used in the proof of the main theorem in §4. §6 contains concluding remarks.

## 2. Model Problem and Preliminaries

We present a model problem and auxiliary results in this section.

**2.1. Mathematical Model.** We consider a time-dependent linear advection-diffusion equation with a degenerate diffusion in one space dimension

$$\begin{aligned} u_t + V(x,t)u_x - (D(x,t)u_x)_x &= f(x,t), \quad (x,t) \in (a,b) \times (0,T), \\ u(x,0) &= u_o(x), \qquad x \in [a,b]. \end{aligned} \tag{1}$$