MODEL ERROR IN THE LANS-ALPHA AND NS-ALPHA DECONVOLUTION MODELS OF TURBULENCE

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Abstract. This paper reports on a computational study of the model error in the LANS-alpha and NS-alpha deconvolution models of homogeneous isotropic turbulence. Computations are also performed for a new turbulence model obtained as a rescaled limit of the deconvolution model. The technique used is to plug a solution obtained from direct numerical simulation of the incompressible Navier–Stokes equations into the competing turbulence models and to then compute the time evolution of the resulting residual. All computations have been done in two dimensions rather than three for convenience and efficiency. When the effective averaging length scale in any of the models is $\alpha_0 = 0.01$ the time evolution of the root-mean-squared residual error grows as $\sqrt{t}$. This growth rate similar to what would happen if the model error were given by a stochastic force. When $\alpha_0 = 0.20$ the residual error grows linearly. Linear growth suggests that the model error possesses a systematic bias. Finally, for $\alpha_0 = 0.04$ the residual error in LANS-alpha model exhibited linear growth; however, for this value of α_0 the higher-order alpha models that were tested did not.

Key words. Alpha-model, bias, model error, Navier-Stokes, stochastic force.

1. Introduction

Consider two dynamical systems

$$\frac{du}{dt} = \mathcal{F}(u) \qquad \text{and} \qquad \frac{dv}{dt} = \widetilde{\mathcal{F}}(v)$$

on a Hilbert space V with norm $\|\cdot\|$. Suppose the evolution of u is given by exact dynamics and the evolution of v according to some approximate dynamics. Define the model error of the approximate dynamics as the residual R obtained by plugging the exact solution u into the equation governing v. Thus,

$$(1) \qquad dR = du - \widetilde{\mathcal{F}}(u)dt = \big(\mathcal{F}(u) - \widetilde{\mathcal{F}}(u)\big)dt$$

where by convention we take $R(0) = 0$. Specifically, consider the case where $\mathcal{F}$ is given by the two-dimensional incompressible Navier–Stokes equations and $\widetilde{\mathcal{F}}$ represents a particular alpha turbulence model. The focus of this paper is whether, to what extent, and under what conditions do the residuals R obtained through numeric computation behave qualitatively as spatially-correlated and temporally-white Gaussian processes.

This question is motivated, in part, by the analysis of Hoang, Law and Stuart [20] for the 4DVAR data assimilation algorithm. That analysis assumes $R = W$ where W is a spatially-correlated and temporally-white Gaussian process and proceeds to show that the inverse problem of finding the initial condition u_0 and the posterior distribution of W is a continuous function of noisy observations of the velocity field. In light of this result, we are interested whether the assumption $R = W$ is realistic when the residual error is given by actual turbulence models. We are also motivated

by the simple desire to compare different turbulence models. Stolz, Adams and Kleiser [33] state that taking the order for the NS-alpha deconvolution models to be $d = 3$ already gives acceptable results while choosing the order larger than 5 does not improve the results significantly. We test this claim by examining the growth rate of R for different values of d and comparing the results to a new rescaled limit of the deconvolution model which has an exponentially small consistency error.

The LANS-alpha model of turbulence is given by the equations

$$(2) \quad \begin{aligned} &\frac{\partial v}{\partial t} + (\bar{v} \cdot \nabla)v + v_j \nabla \bar{v}_j = \nu \Delta v - \nabla p + f, \\ &\nabla \cdot \bar{v} = 0 \qquad \text{where} \qquad v = (1 - \alpha^2 \Delta)\bar{v}. \end{aligned}$$

Here v is the Eulerian velocity field, $\bar{v}$ is the average velocity field, α is the averaging length scale, ν is the kinematic viscosity, p is the physical pressure and f is a body force. Note that setting $\alpha = 0$ yields the standard Navier–Stokes equations. These equations, originally called the viscous Camassa–Holm equations, were introduced as a closure for the Reynolds averaged Navier–Stokes equations by Chen, Foias, Holm, Olson, Titi and Wynne in 1998 through a series of papers [4, 5, 6]. At the same time, numerical simulations by Chen, Holm, Margolin and Zhang [7] concluded that the LANS-alpha model also functions as an effective subgrid-scale model. Connections to the theory of global attractors and homogeneous isotropic turbulence and global attractors appear in [14].

Note that equations (2) can be derived as the Euler-Poincaré equations of an averaged Lagrangian to which a viscous term, obtained by identifying the momentum in the physical derivation, has been added. This derivation further assumes that the turbulence is homogeneous and isotropic. A body of theoretical and numerical literature on the LANS-α model exists—see [3, 8, 10, 18, 20, 21, 24, 25, 26] and references therein—that, among other things, explores the dependency on α and the limit when $\alpha \to 0$, relaxes the homogeneity and isotropy assumptions, studies boundary conditions and boundary layers, and treats other physical systems. In summary, the LANS-alpha model is a well-studied turbulence model that is suitable for further study here.

To avoid a study of boundary layers we consider flows in domains with periodic boundary conditions. To approximate homogeneous and isotropic turbulence we choose a time-independent body forcing that has no regular patterns in space and for which the resulting flow undergoes complex time dependent behavior that in no way resembles the force, see Figure 3. Since the body-force is time independent, it is natural to suppose the statistics of the flow are stationary. While these assumptions are consistent with the classical theories of fully developed turbulence developed by Kolmogorov [29] and Kraichnan [30], the possibility of intermittency may lead to non-equilibrium and non-stationarity. Moreover, while domains with periodic boundary conditions are obviously homogeneous, the presence of any non-zero forcing function has the potential to render the statistics of the resulting flow inhomogeneous. As noted by Kurien, Aivalis and Sreenivasan [31], see also Taylor, Kurien and Eyink [34], even when the body forcing is zero, turbulent flows in periodic domains can possess a certain degree of anisotropy. It is hoped, therefore, that the stationarity, homogeneity and isotropy assumptions made in the derivation of the LANS-alpha and NS-alpha deconvolution models are well enough satisfied that the turbulence models studied here apply. Viewed in a different way, our computations of the residual error may be seen as a test of these assumptions.