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NUMERICAL ANALYSIS FOR A NONLOCAL ALLEN-CAHN EQUATION

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Abstract. We propose a stable, convergent finite difference scheme to solve numerically a nonlocal Allen-Cahn equation which may model a variety of physical and biological phenomena involving long-range spatial interaction. We also prove that the scheme is uniquely solvable and the numerical solution will approach the true solution in the L^∞ norm.

Key Words. Finite difference scheme; Long range interaction.

1. Introduction

Consider the following problem

$$u_t = \int_\Omega J(x-y)u(y)dy - \int_\Omega J(x-y)dy\, u(x) - f(u) \tag{1}$$

in $(0,T)\times\Omega$, with initial condition

$$u(0,x) = u_0(x), \tag{2}$$

where $T>0$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. The unknown u is a real-valued order parameter, the interaction kernel satisfies $J(-x)=J(x)$, and f is bistable.

The equation (1) can be derived as an L^2 gradient flow for the free energy

$$E = \frac{1}{4}\int\int J(x-y)\left(u(x)-u(y)\right)^2 dx\,dy + \int F\left(u(x)\right)dx, \tag{3}$$

where F is a double well function.

The L^2 gradient flow for the classical Ginzberg-Landau energy functional

$$E = \frac{1}{2}\int |\nabla u|^2 dx + \int F\left(u(x)\right)dx, \tag{4}$$

is the Allen-Cahn equation:

$$u_t = \triangle u(x) - f(u) \tag{5}$$

As mentioned in [3], the equations (1) and (5) are important for modelling a variety of physical and biological phenomena involving media with properties varying in space. There is by now a lot of work on equation (1) and (5) (see for example [1], [2], [5], [7], [8], [9], [11], [12], [13], [15], [16], [17], and the references therein).

To the best of our knowledge, there are very few results on the numerical solutions to (1). In this paper, we develop a finite difference scheme for equation (1) for $n=1$ and $n=2$. We also prove that the difference scheme is stable and that the numerical

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approximation converges to the solution of (1) as the spatial and temporal mesh size approaches zero. Our numerical results coincide with the theoretical results in [12].

2. Analysis of the proposed scheme

In this section, we consider finite difference approximations of equation (1) for $n = 1$ and $n = 2$. For the sake of exposition, we take $f(u) = u^3 - u$, but the analysis applies to the general smooth bistable function if care is taken in the choice of linearization.

We use the following notation:

For $n = 1$ with $\Omega = (-L, L)$,

$$\Omega_x = \{x_i|\, x_i = -L + i\triangle x,\, 0 \le i \le M\},$$
$$\Omega_t = \{t_k|\, t_k = k\triangle t,\, 0 \le k \le K\},$$

where $\triangle x = 2L/M$ and $\triangle t = T/K$. Our difference scheme for equation (1) for $n = 1$ is as follows:

$$u_i^0 = u_0(x_i), \text{ for } 0 \le i \le M, \tag{6}$$

$$\delta_t u_i^k = (J * u^k)_i - (J * 1)_i u_i^k + \psi(u_i^k, u_i^{k+1}) \text{ for } 0 \le i \le M,\, 0 \le k \le K-1, \tag{7}$$

where

$$\delta_t u_i^k = \frac{u_i^{k+1} - u_i^k}{\triangle t},$$

$$(J * u^k)_i = \triangle x \left[\frac{1}{2} J(x_0 - x_i) u_0^k + \sum_{m=1}^{M-1} J(x_m - x_i) u_m^k + \frac{1}{2} J(x_M - x_i) u_M^k \right],$$

and

$$\psi(u_i^k, u_i^{k+1}) = u_i^k - (u_i^k)^2 u_i^{k+1}.$$

For a rectangular domain $(-L, L) \times (-W, W) \subset \mathbb{R}^2$, we have

$$\Omega_{x,y} = \{(x_i, y_j)|\, x_i = -L + i\triangle x,\, y_j = -W + j\triangle y,\, 0 \le i \le M,\, 0 \le j \le N\},$$
$$\Omega_t = \{t_k|\, t_k = k\triangle t,\, 0 \le t \le K\},$$

where $\triangle x = 2L/M$ and $\triangle y = 2W/N$.

Our difference scheme in this case is

$$u_{i,j}^0 = u_0(x_i, y_j) \text{ for } 0 \le i \le M,\, 0 \le j \le N, \tag{8}$$

$$\delta_t u_{i,j}^k = (J * u^k)_{i,j} - (J * 1)_{i,j} u_{i,j}^k + \psi(u_{i,j}^k, u_{i,j}^{k+1}) \quad \text{for } 0 \le i \le M,\, 0 \le j \le N,\, 0 \le k \le K-1, \tag{9}$$

where

$$\delta_t u_{i,j}^k = \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\triangle t},$$