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A DOMAIN DECOMPOSITION METHOD WITH LAGRANGE MULTIPLIER BASED ON THE POINTWISE MATCHING CONDITION

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Abstract. In this paper, we are concerned with a non-overlapping domain decomposition method with nonmatching grids. In this method, a new pointwise matching condition is used to define weak continuity of approximate solutions on the interface. The main merit of the new method is that numerical integrations can be avoided when calculating interface matrices. We derive an almost optimal error estimate of the resulting approximate solutions for two kinds of applicable situations. Some numerical experiments confirm the theoretical result.

Key Words. domain decomposition, nonmatching grids, pointwise matching, error estimate.

1. Introduction

The domain decomposition method (DDM) with nonmatching grids is now popular in engineering and scientific computing (see [1], [2], [5], [9], [10], [12], [13], [14], [15] and [17]). A key ingredient in this method is the choice of a suitable interface matching condition, which defines the discrete variational problem associated with this DDM. The convergence of the resulting approximate solution, which is only weak continuous across the interface, strongly depends on such interface matching condition.

There are two kinds of interface matching conditions in literature: the integral matching condition (see [4], [3], [6] and [13]), and the pointwise matching condition (see [4] and [6]). When using the integral matching condition, calculation of numerical integrations on the interface will be in general expensive, especially for three-dimensional problems. Use of the pointwise matching condition can remove this difficulty, but it may generate unsatisfactory approximate solutions (see [4] and [6]).

In the present paper we investigate when the pointwise matching condition works well in DDM with nonmatching grids for the second-order elliptic problem in three dimensions. The main difficulty is the design of a weak conformity on the *wire-basket* set to the approximation. For two-dimensional problems, one can require that the approximation is continuous at the cross-points (see [4]). But, one can not impose the same continuity on the *wire-basket* set, since the grids on the *wire-basket*

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set are still nonmatching. If no constraint is added on the *wire-basket* set, the resulting approximation has low convergence. To remove this difficulty, we propose a combination between the pointwise matching condition and the integral matching condition. The idea is to define a suitable *discrete* L^2 projection into the Lagrange multiplier space defined on the common face of two neighboring subdomains. The new matching condition implies that the restrictions of the underlying approximation on two neighboring subdomains has the same *discrete* L^2 projection. In essence, the new matching condition involves a set of internal nodes on each local face. We require that the approximation has point to point continuity at the nodes not closing the boundary of the face, and possesses weak continuity in the sense of average at the nodes closing the boundary of the face. The whole matching condition can be expressed in a unified manner by defining an interpolation type operator. It will be shown that the resulting approximate solution possesses almost the optimal error estimate for two practical situations.

The outline of this paper is as follows. In Section 2, we introduce DDM with the new pointwise matching condition. In Section 3, we show that this pointwise matching condition can result in almost the optimal error estimate for two applied cases. In Section 4, we give some numerical results, which confirm the effectiveness of this new interface matching condition.

2. DDMs with Pointwise Matching Condition

In this paper, we consider the following model problem

$$\left\{ \begin{aligned} -\nabla(\omega\nabla u) &= \quad f, \quad in\ \Omega, \\ u &= \quad 0, \quad on\ \partial\Omega, \end{aligned} \right. \tag{1}$$

where $\Omega \subset R^3$ is a bounded, connected Lipschitz domain, and $\omega \in L^\infty(\Omega)$ is a positive real function.

Let $H_0^1(\Omega)$ be the standard Sobolev space and define the following bilinear form:

$$a(u,v) = \int\limits_\Omega \omega\nabla u\cdot\nabla v dx,\ u,v\in H_0^1(\Omega).$$

Then the corresponding weak form of (1) is: Find $u \in H_0^1(\Omega)$, such that:

$$a(u,v) = (f,v),\ \forall\ v\in H_0^1(\Omega), \tag{2}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product.

In the following, we define the discrete problem of (2) based on DDMs with a new pointwise matching condition.

As usual, we decompose the domain Ω into the union of some subdomains $\overline{\Omega} = \sum\limits_{k=1}^{N} \overline{\Omega}_k$, which satisfies that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. For convenience, we assume that the decomposition is geometrically conforming:

(1) if Ω_i and Ω_j are two neighboring subdomains, then $\partial\Omega_i \cap \partial\Omega_j$ is just a common vertex, or a common edge or a common face of Ω_i and Ω_j.

(2) each subdomain has the same "size" d in the usual way (refer to [20]).

In particular, when $\partial\Omega_i \cap \partial\Omega_j$ is a common face of Ω_i and Ω_j, we set $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and call Γ_{ij} to be a local interface.

For each Ω_k, we introduce a partition $\mathcal{T}_k$ which is made of elements that are either hexahedra or tetrahedra. Let h_k be the mesh size of $\mathcal{T}_k$, i.e., h_k denotes the maximum diameter of all elements in $\mathcal{T}_k$. Define $h = \min\limits_{1\le k\le N} h_k$. The triangulation