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# CONVERGENCE OF THE TIME-DOMAIN PERFECTLY MATCHED LAYER METHOD FOR ACOUSTIC SCATTERING PROBLEMS

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**Abstract.** In this paper we establish the stability and convergence of the time-domain perfectly matched layer (PML) method for solving the acoustic scattering problems. We first prove the well-posedness and the stability of the time-dependent acoustic scattering problem with the Dirichlet-to-Neumann boundary condition. Next we show the well-posedness of the unsplit-field PML method for the acoustic scattering problems. Then we prove the exponential convergence of the non-splitting PML method in terms of the thickness and medium property of the artificial PML layer. The proof depends on a stability result of the PML system for constant medium property and an exponential decay estimate of the modified Bessel functions.

**Key Words.** perfectly matched layer, acoustic scattering, exponential convergence, stability

## 1. Introduction

We consider the acoustic scattering problem with the sound-hard boundary condition on the obstacle

$$\frac{\partial u}{\partial t} = -\mathrm{div}\mathbf{p} + f(\mathbf{x},t), \qquad \frac{\partial \mathbf{p}}{\partial t} = -\nabla u \quad \text{in } [\mathbb{R}^2\backslash\bar{D}]\times(0,T), \tag{1.1}$$

$$\mathbf{p}\cdot\mathbf{n}_D = 0 \quad \text{on } \Gamma_D\times(0,T), \tag{1.2}$$

$$\sqrt{r}(u-\mathbf{p}\cdot\hat{\mathbf{x}}) \to 0, \quad \text{as } r=|\mathbf{x}|\to\infty, \quad \text{a.e. } t\in(0,T), \tag{1.3}$$

$$u|_{t=0} = u_0, \quad \mathbf{p}|_{t=0} = \mathbf{p}_0. \tag{1.4}$$

Here $u$ is the pressure and $\mathbf{p}$ is the velocity field of the wave. $D\subset\mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\Gamma_D$, $\hat{\mathbf{x}}=\mathbf{x}/|\mathbf{x}|$, and $\mathbf{n}_D$ is the unit outer normal to $\Gamma_D$. $f, u_0, \mathbf{p}_0$ are assumed to be supported in the circle $B_R=\{\mathbf{x}\in\mathbb{R}^2 : |\mathbf{x}|<R\}$ for some $R>0$. (1.3) is the radiation condition which corresponds to the well-known Sommerfeld radiation condition in the frequency domain. We remark that the results in this paper can be easily extended to solve scattering problems with other boundary conditions such as the sound-soft or the impedance boundary condition on $\Gamma_D$.

One of the fundamental problems in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. The first objective of this paper is to prove the well-posedness and stability of the system (1.1)-(1.4) by imposing the

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Dirichlet-to-Neumann boundary condition on the $\Gamma_R = \partial B_R$. The proof depends on the abstract inversion theorem of the Laplace transform and the a priori estimate for the Helmholtz equation which seems to be new and is of independent interest. In Lax and Phillips [20], the scattering problem of the wave equation is studied by using the semigroup theory of operators in the absence of the source function $f$. We remark that the well-posedness of scattering problems in the frequency domain is well-known (cf. e.g. Colten and Kress [10]).

The non-local Dirichlet to Neumann boundary condition for (1.1)-(1.4) is the starting point of various approximate absorbing boundary conditions which have been proposed and studied in the literature, see the review papers Givoli [16], Tsynkov [25], Hagstrom [17] and the references therein. An interesting alternative to the method of absorbing boundary conditions is the method of perfectly matched layer (PML). Since the work of Bérenger [5] which proposed a PML technique for solving the time-dependent Maxwell equations in the Cartesian coordinates, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [27], Teixeira and Chew [24] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain.

There are two classes of time-domain PML methods for the wave scattering problems. The first class, called “split-field PML method” in the literature, includes the original Bérenger PML method. It is shown in Abarbanel and Gottlieb [2] that the Bérenger PML method is only weakly well-posed and thus may suffer instability in practical applications. The second class, the so-called “unsplit-field PML formulations” in the literature, is however, strongly well-posed. One such successful method is the uniaxial PML method developed in Sacks *et al* [23] and Gedney [15] for the Maxwell equations in the Cartesian coordinates. In the curvilinear coordinated, the split-field PML method is introduced in Collino and Monk [9] and the unsplit-field PML methods are introduced in Petropoulos [22] and [24] for Maxwell equations.

Although the tremendous attention and success in the application of PML methods in the engineering literature, there are few mathematical results on the convergence of the PML methods. For the Helmholtz equation in the frequency domain, it is proved in Lassas and Somersalo [19], Hohage *et al* [18] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. In Chen and Wu [8], Chen and Liu [7], an adaptive PML technique is proposed and studied in which a posteriori error estimate is used to determine the PML parameters. In particular, it is shown that the exponential convergence can be achieved for fixed thickness of the PML layer by enlarging PML medium properties. For the time-domain PML method, not much mathematical convergence analysis is known except the work in Hagstrom [17] in which the planar PML method in one space direction is considered for the wave equation. In de Hoop *et al* [12], Diaz and Joly [13], the PML system with point source is analyzed based on the Cagniard - de Hoop method.

The long time stability of the PML methods is also a much studied topic in the literature (see e.g. Bécache and Joly [3], Bécache *et al* [4], Appelö *et al* [1]). For a PML method to be practically useful, it must be stable in time, that is, the solution should not grow exponentially in time. We remark that the well-posedness of the