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# A COMPUTATIONAL SCHEME FOR OPTIONS UNDER JUMP DIFFUSION PROCESSES

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**Abstract.** In this paper we develop two novel numerical methods for the partial integral differential equation arising from the valuation of an option whose underlying asset is governed by a jump diffusion process. These methods are based on a fitted finite volume method for the spatial discretization, an implicit-explicit time stepping scheme and the Crank-Nicolson time stepping method. We show that the discretization methods are unconditionally stable in time and the system matrices of the resulting linear systems are $M$-matrices. The resulting linear systems involve products of a dense matrix and vectors and an Fast Fourier Transformation (FFT) technique is used for the evaluation of these products. Furthermore, a splitting technique is proposed for the solution of the discretized system arising from the Crank-Nicolson scheme. Numerical results are presented to show the rates of convergence and the robustness of the numerical method.

**Key Words.** Jump diffusion processes, Option pricing, Finite volume method, Integral partial differential equation, FFT.

## 1. Introduction

It is well known that the assumption of log-normal stock diffusion with constant volatility in the standard Black-Scholes model of option pricing is not consistent with that of the market price movement. This phenomenon is often referred to as the volatility skew or smile [10] and exists in all the major stock index markets today. In order to capture the existence of volatility smiles, extensions of the Black-Scholes model have been proposed. Generally speaking, three approaches have been studied in the finance literature: the stochastic volatility approach [9, 11], the deterministic volatility function approach [8] and the jump diffusion model [14, 23, 5, 7]. Among them, jump diffusion , first introduced by Merton in [14], is more attractive than the other two. Contrary to the Black-Scholes model [4], the stock price in the jump diffusion model is not a continuous function of time. This allows to account for large changes in market prices due to rare events. More importantly, the jump diffusion model yields implied volatility curves similar to volatility smiles observed on markets.

Unlike the standard Black-Scholes model, the valuation of options under jump diffusion processes requires solving a partial integral differential equation. This is challenging to handle numerically since a non-local integration term is involved. There are several existing numerical methods based on the finite difference method for this problem. In [2], a method based on the multinomial trees is proposed, which is actually an explicit type finite difference approach. Hence, it is first-order accurate and conditionally stable. In [23] the author developed a method for the equation which treats the integral term explicitly and the other terms implicitly.

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This method is only first-order accurate and conditionally stable. In [3], an operator splitting method coupled with an FFT for the evaluation of the integral term is proposed, producing an unconditionally stable and 2nd-order accurate scheme. In [1], a second order backward difference scheme in time is developed, where the FFT technique is also used to evaluate the integral term and two operator splitting methods are proposed to solve the resulting system iteratively.

It is well known that in the case that the volatility or underlying asset price goes to zero, the Black-Scholes partial differential equation becomes convection-dominant so that solutions to the equation may display boundary or interior layers. Standard methods such as the central finite difference and piecewise linear finite element methods cannot handle this difficulty [19]. The same problem also appear in the partial integral differential equation resulting from the jump diffusion model. To overcome this difficulty, a fitted finite volume method is designed in [18, 20] to price the European and American options. The method is based on a popular exponentially fitting technique widely used for problems with boundary and interior layers (cf. [15, 16]).

In this paper, we present two discretization methods for the partial integral differential equation arising from the valuation of European vanilla options. Although the methods are presented for this particular option, they can easily be extended to other option pricing problems. The methods are based on the fitted finite volume method for spatial discretization [18], an implicit-explicit time stepping method and Crank-Nicolson time discretization scheme, coupled with FFT for the evaluation of the integral term. The Crank-Nicolson method results in a dense system matrix. To avoid the inversion of the dense matrix, we develop an iterative algorithm to solve the resulting system, based on a regular operator splitting. We prove that both of the numerical methods are unconditionally stable and their system matrices are both $M$-matrices. Numerical experiments are performed using Merton's model [14] and Kou's model [12]. Numerical results show that the methods are of 1st- and 2nd-order accuracy, respectively, and are robust.

The paper is organized as follows. In the next section, the mathematical model for pricing options with jump diffusion processes is presented. In Section 3, the fitted finite volume method is developed for the equation. A full discretization is proposed in Section 4 in which a stability and convergence theory for the method is also established. Also in this section, an algorithm for the numerical solution of the discretized system is proposed. Finally, in Section 5 we present some numerical results to demonstrate the convergence rates and robustness of the numerical schemes.

## 2. The pricing model

Let $S$ denote the price of an asset and assume its movement follows the jump diffusion dynamics described by follow the following stochastic differential equation

$$\frac{dS}{S} = (\nu - \lambda\kappa)\,dt + \sigma dZ + (\eta - 1)\,dq, \tag{1}$$

where $dZ$ is an increment of the standard Gauss-Wiener process and $dq$ is the independent Poisson process with a deterministic jump intensity $\lambda$. Also in (1), $\sigma$ is the volatility, $\upsilon$ is the drift rate; and $\eta - 1$ is an impulse function producing a jump from $S$ to $S\eta$, and $\kappa = E(\eta - 1)$, where $E(\cdot)$ denotes the expectation operator.

Let $V(S,t)$ denote the value of a European contingent claim with striking price $K$ on the underlying asset $S$ and time $t$. By a standard argument (cf., for example, [21]), it is easy to show that $V(S,\tau)$ satisfies the following backward partial integral