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A PRE–PROCESSING MOVING MESH METHOD FOR DISCONTINUOUS GALERKIN APPROXIMATIONS OF ADVECTION–DIFFUSION–REACTION PROBLEMS

PAOLA F. ANTONIETTI AND PAUL HOUSTON

Abstract. We propose a pre–processing mesh re-distribution algorithm based upon harmonic maps employed in conjunction with discontinuous Galerkin approximations of advection–diffusion–reaction problems. Extensive two–dimensional numerical experiments with different choices of monitor functions, including monitor functions derived from goal–oriented *a posteriori* error indicators are presented. The examples presented clearly demonstrate the capabilities and the benefits of combining our pre–processing mesh movement algorithm with both uniform, as well as, adaptive isotropic and anisotropic mesh refinement.

Key Words. Discontinuous Galerkin methods, advection–diffusion problems, adaptivity.

1. Introduction

The modeling of the interaction between advective and diffusive processes is of fundamental importance in many areas of applied mathematics. Typically, in applications, advection essentially dominates diffusion, which leads to a 'nearly' hyperbolic set of governing partial differential equations. Moreover, solutions to these equations exhibit localized phenomena, such as propagating 'near–shocks' and sharp transition layers, and their numerical approximation presents a challenging computational task; indeed, it is well documented that many standard numerical methods, developed for diffusion–dominated processes, often behave very poorly when applied to these types of problems. Additionally, the presence of local singularities in the solution may lead to a global deterioration of the numerical approximation. Indeed, when uniform meshes are employed, the computational cost to obtain accurate numerical solutions is typically very high, particularly for three-dimensional problems. Therefore, the development of effective and robust adaptive methods for these types of problems becomes a computational necessity. The successful implementation of adaptive strategies, on the one hand, can increase the accuracy of the numerical approximation and, on the other hand, decrease the computational cost. The adaptive strategies developed within the context of finite element methods, can be broadly classified as follows:

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h-method: This involves the automatic refinement or coarsening of the computational mesh based on suitable *a posteriori* error estimates or error indicators;

p-method: This involves the enrichment of the local (elemental) polynomial degree;

hp-method: This combines both local h– and p–refinement based on a local decision taken on each element of the computational mesh as to which refinement strategy (*i.e.*, h–refinement or p–refinement) should be employed on the element in order to obtain the greatest reduction in the error per unit cost. It exploits suitable control techniques which assess the local regularity of the underlying analytical solution; for example, one may determine the analyticity of a function by writing it in terms of a convergent Legendre series expansion, and assessing the rate at which the Legendre coefficients tend to zero, cf. [18, 8];

r-method (moving mesh method): This approach *relocates* (without affecting the mesh topology) the grid points of a mesh, keeping the number of nodes fixed, in such a way that the nodes become concentrated in regions of the computational domain where the analytical solution undergoes rapid variation.

On the one hand, considerable progress has been made on both the *a posteriori* error analysis of finite element methods for a wide range of partial differential equations of practical interest, and the development of reliable and robust automatic h–, p– and hp–strategies (see, for example, [1, 4, 9, 25, 26, 28, 8], and the references therein). On the other hand, the state of development of "optimal" mesh modification strategies which are capable of delivering the greatest reduction in the error for the least amount of computational cost, is far less advanced.

In recent years considerable work has been devoted to the development of r–adaptive finite element algorithms, which, for a fixed polynomial order at least, seek to re-distribute the nodes of a given mesh in an optimal fashion; see, for example, [29, 23, 21, 23, 20, 19, 22], and the references cited therein. The moving mesh method is very well suited for dynamical problems, and indeed problems with moving boundaries, though such approaches may also be employed to optimize a mesh for a stationary PDE by employing a nonlinear iteration similar to that employed in h–adaptive methods. An r–refinement method usually contains two key steps: a mesh selection algorithm and a solution algorithm. In some of the existing r-methods, these two parts are strongly associated with each other, and any change of the underlying partial differential equation will result in the rewriting of large parts of the computational code. The success of a mesh adaptation strategy using a variational approach hinges on choosing an appropriate monitor function (cf. [6], for a study of this aspect of the adaptive mesh generation problem). For example, for linear finite elements, the monitor function is often given in terms of some first–/second–order derivatives of the computed solution.

One major drawback of r–refinement techniques is that they are often very expensive, particularly when the underlying mesh is extremely fine; in this case any iterative approach employed to move the mesh may converge very slowly, if at all. In this article, we aim exploit the numerous advantages of the original r-method based on harmonic maps to develop an optimal pre–processing algorithm to be employed in conjunction with discontinuous Galerkin (DG, for short) approximations