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CONVERGENCE ANALYSIS OF A SPLITTING METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we propose a fully drift-implicit splitting numerical scheme for the stochastic differential equations driven by the standard d-dimensional Brownian motion. We prove that its strong convergence rate is of the same order as the standard Euler-Maruyama method. Some numerical experiments are also carried out to demonstrate this property. This scheme allows us to use the latest information inside each iteration in the Euler-Maruyama method so that better approximate solutions could be obtained than the standard approach.

Key Words. Stochastic differential equation, drift-implicit splitting scheme, Brownian motion

1. Introduction

Let us consider the following stochastic differential equations (SDEs)

$$\begin{cases} dy(t) = f(y(t))dt + g(y(t))dW(t), & 0 \le t \le T \\ y(0) = y_0 \end{cases} \tag{1}$$

where $T > 0$ is the terminal time, $y(t) : [0, T] \times \Omega \to R^m$, $f(y) : R^m \to R^m$, $g(y) : R^m \to R^{m\times d}$, and $W(t) = (W_1(t), \cdots, W_d(t))^*$ is a standard d-dimensional Brownian motion defined on a complete, filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\le t\le T})$. Stochastic differential equations are used in many fields, such as stock market, financial mathematics, stochastic controls, dynamic system, biological science, chemical reactive kinetics and hydrology, and so on. Thus, it is of importance to study the solution of SDEs. However, it is often very difficult or impossible to find the analytic solutions of SDEs, as a consequence, numerical methods for finding approximate solutions of SDEs have attracted much attentions.

There have been a lot of publications in which numerical methods for stochastic differential equations and their applications were studied and discussed. For instance, the Itô-Taylor type method proposed in [11] that makes use of the so-called Itô Taylor expansion to discretize the SDEs; the linearization type methods suggested in [3, 12, 17], that first linearize the drift and diffusion coefficients of the SDEs and then solve the pruned linear SDEs instead; the Runge-Kutta type methods [4,5,16,20], in which the Runge-Kutta methods for solving ordinary differential equations are extended to solve the SDEs. Concerning the stability of the methods,

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some implicit discretization schemes were proposed in [6, 7] to stablize the numerical discretization. In order to improve the accuracy of the approximate solution, some high-order numerical methods for solving SDEs were studied in [1, 4, 5, 10, 11] and some splitting methods were also studied in [2].

The Euler-Maruyama (E-M) method is so far the most studied numerical method for solving SDEs and its strong convergence rate is 1/2 for general cases. Due to its easy implementation, the E-M method and its modified versions have been very commonly used for applied stochastic problems, such as stochastic optimal control and stochastic partial differential equations. Since SDEs are often driven by a high-dimensional Brownian motion and coupled with other type stochastic problems [9], more efficient and accurate solvers for high-dimensional SDEs are urgently needed. In the past decades, the operator splitting scheme has been extensively studied and becomes one of the most popular and efficient ways to deal with multi-dimensional problems which are modeled by the deterministic ordinary or partial differential equations. In fact, the same idea also can be applied to the SDEs. In this paper, we will propose a new splitting scheme for numerical solutions of the SDEs (1), and show that the resulted approximate solution converges to the analytic solution of the SDEs with the same convergence rate as the one the E-M method has. Furthermore, this scheme allows us to use the latest information inside each iteration in the E-M method so that better approximate solutions could be obtained than the standard approach especially when d is large.

We organize this paper as follows. In Section 2, we first propose a fully drift-implicit splitting scheme for the discretization of the SDEs (1), then we prove the strong convergence of this scheme in Section 3. After presenting some computational experiments in Section 4, conclusions are given in Section 5.

2. A fully drift-implicit splitting scheme of SDEs

Let us rewrite the stochastic differential equations (1) in the following form:

$$\begin{cases} dy(t) &= f(y(t))dt + \sum_{i=1}^{d} g_i(y(t))dW_i(t) \qquad ,0 < t \leq T \\ y(0) &= y_0, \end{cases} \tag{2}$$

where $W_i(t)$, $i = 1, 2, \cdots, d$ are independent one-dimensional Brownian motions and $g_i : R^m \to R^m$, $i = 1, 2, \cdots, d$.

It is well-known that the problem (2) is equivalent to the following Itô integral equation

$$y(t) \quad = \quad y_0 + \int_0^t f(y(s))ds + \sum_{i=1}^{d} \int_0^t g_i(y(s))dW_i(s). \tag{3}$$

To discretize the equation (2), we first partition the time interval $[0, T]$ by

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T. \tag{4}$$

Let $\Delta t_n = t_{n+1} - t_n$ denote the discrete time step at the time t_n, and set $\Delta t = \max_{n=0}^{N-1} \Delta t_n$. For the simplicity of description, we only discuss the case of uniform time partition, but all results obtained in this paper still remain valid for general partition (4).

From the equation (3), we have exactly

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(y(s))ds + \sum_{i=1}^{d} \int_{t_n}^{t_{n+1}} g_i(y(s))dW_i(s). \tag{5}$$