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OPTIMAL DESIGN OF THE SUPPORT OF THE CONTROL FOR THE 2-D WAVE EQUATION: A NUMERICAL METHOD

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Abstract. We consider in this paper the homogeneous 2-D wave equation defined on $\Omega \subset \mathbb{R}^2$. Using the Hilbert Uniqueness Method, one may associate to a suitable fixed subset $\omega \subset \Omega$, the control v_ω of minimal $L^2(\omega \times (0,T))$-norm which drives to rest the system at a time $T > 0$ large enough. We address the question of the optimal position of ω which minimize the functional $J : \omega \to ||v_\omega||_{L^2(\omega\times(0,T))}$. Assuming $\omega \in C^1(\Omega)$, we express the shape derivative of J as a curvilinear integral on $\partial\omega \times (0,T)$ independently of any adjoint solution. This expression leads to a descent direction and permits to define a gradient algorithm efficiently initialized by the topological derivative associated to J. The numerical approximation of the problem is discussed and numerical experiments are presented in the framework of the level set approach. We also investigate the well-posedness of the problem by considering its relaxation.

Key Words. Optimal shape design, Exact controllability of wave equation, Level set method, Numerical schemes, Relaxation.

1. Introduction - Problem statement

We consider in this work a general optimal design problem in the context of the (exact) controllability. There is by now a large interest in optimal shape design theory [9, 14], specially for dynamical system [13, 24], which consists in optimizing the distributions of materials or the shape of a mechanical structure in order to reach a suitable optimal behavior with respect to some initial excitation. On the other hand, since twenty years, a huge literature in the field of control has been devoted to the modeling and the analysis of mechanical systems, stabilized or exactly controlled in time, by some boundary or internal dissipative mechanisms [16, 17, 18, 19]. For instance, we mention the example of a multi-layered composite plate locally controlled by some piezo-electric device [15]. In order to extend this optimization process, it appears natural to optimize the shape and design of such dissipative mechanisms, distributed on the structure. We treat here this question in the context of the 2-D wave equation with an internal control. To the knowledge of the author, the coupling of these two notions - shape optimal design and exact controllability - has not been addressed so far.

Let us consider a Lipschitzian bounded domain $\Omega \in \mathbb{R}^2$, two functions $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and a real $T > 0$. In the context of the exact distributed controllability, one may determine a subset ω of positive Lebesgue measure for which the following property holds (see [3, 12, 19]) : there exists a control function $v_\omega \in$

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$L^2(\omega\times(0,T))$ such that the unique solution $y \in C([0,T]; H_0^1(\Omega))\cap C^1([0,T]; L^2(\Omega))$ of

$$\begin{cases} y_{tt} - \Delta y = v_\omega \mathcal{X}_\omega, & \Omega\times(0,T), \\ y = 0, & \partial\Omega\times(0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (y^0, y^1), & \Omega, \end{cases} \tag{1}$$

satisfies

$$y(.,T) = y_t(.,T) = 0, \quad \text{on} \quad \Omega. \tag{2}$$

y_t denotes the derivative of y with respect to t and $\mathcal{X}_\omega \in L^\infty(\Omega, \{0,1\})$ denotes the characteristic function of the subset ω. We introduce the set

$$V(y^0, y^1, T) = \{\omega \subset \Omega \text{ such that (2) holds}\} \tag{3}$$

which contains in particular Ω. Moreover, from [3] assuming $\Omega \in C^\infty(\mathbb{R}^2)$, any subset ω satisfying the geometric control condition in Ω (*"Every ray of geometric optics that propagates in Ω and is reflected on its boundary enters ω in time less than T"*) belongs to $V(y^0, y^1, T)$. On the other hand, if Ω is rectangular and T large enough (dependent of the diameter of $\Omega\backslash\omega$), then any domain ω is in $V(y^0, y^1, T)$ (see [12]).

The control problem formulated above is usually referred to as internal (or distributed) controllability problem. The controllability property may be obtained using the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions in [19], which reduces the problem to an optimal control one. Precisely, for any $\omega \in V(y^0, y^1, T)$, the unique control v_ω of minimal L^2-norm (refereed as the HUM control in the sequel) may be obtained by minimizing the functional $\mathcal{J} : L^2(\Omega)\times H^{-1}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}(\phi^0, \phi^1) = \frac{1}{2}\int_\omega\int_0^T \phi^2(\boldsymbol{x}, t)dtdx + < \phi_t(\cdot,0), y^0 >_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega y^1\phi(\cdot,0)dx, \tag{4}$$

where $< \cdot,\cdot >_{H^{-1},H_0^1}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ and ϕ the solution of the adjoint homogeneous system

$$\begin{cases} \phi_{tt} - \Delta\phi = 0, & \Omega\times(0,T), \\ \phi = 0, & \partial\Omega\times(0,T), \\ (\phi(\cdot,T), \phi_t(\cdot,T)) = (\phi^0, \phi^1), & \Omega. \end{cases} \tag{5}$$

This provides the following characterization of the HUM-control (see [19], chapter 7).

THEOREM 1.1. *Given any $(y^0, y^1) \in H_0^1(\Omega)\times L^2(\Omega)$, $T > 0$ and $\omega \in V(y^0, y^1, T)$, the functional $\mathcal{J}$ has a unique minimizer $(\hat\phi^0, \hat\phi^1) \in L^2(\Omega)\times H^{-1}(\Omega)$. If $\hat\phi$ is the corresponding solution of (5) with initial data $(\hat\phi^0, \hat\phi^1)$ then $v = -\hat\phi\mathcal{X}_\omega$ is the control of (1) with minimal L^2-norm.*

This result is based on the following observation or observability inequality (leading to the coercivity of $\mathcal{J}$ in $L^2(\Omega)\times H^{-1}(\Omega)$): there exists a constant $C_{T,\omega} > 0$ function of T and ω (called the observability constant) such that

$$||(\phi^0, \phi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)} \le C_{T,\omega}\int_\omega\int_0^T \phi^2(\boldsymbol{x}, t)dt\, dx \tag{6}$$