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FOUNDATION OF FAST NON-LINEAR FINITE ELEMENT SOLVERS, PART II

PETER L. SHI

Abstract. The author establishes a finite element solver algorithm of optimal speed for a class of quasi-linear equations with large stiffness variations and oscillations. In particular, the algorithm can successfully handle soft inclusions of negative stiffness. Besides the convergence analysis, large number of numerical examples are presented.

Key Words. Finite elements, non-linear solver algorithm, optimal speed

1. Introduction

This is the first of a series of papers supplementing the long article of the author [11] which has established a general algorithmic architecture for solving nonlinear finite element models with linear speed. The focus here is to demonstrate a particular implementation of the methodology to handle the Galerkin formulation of linear and nonlinear finite element models with large stiffness variation and oscillation that frequently arise from composite materials. After a briefing on the general theory and algorithm, we center our discussion around two benchmark problems. The first is concerned with the elasto-plastic deformation of a membrane in which the Young's modulo in the elastic region greatly exceeds that in the plastic region, constituting large jumps in coefficients in unknown regions. The second case is concerned with soft inclusions typically seen in making a composite, in which the included soft material is distributed as mesoscale tiny blocks of much softer stiffness in the scale of $10^{-2} \sim 10^{-6}$ compared to the hard material matrix. In particular, we demonstrate the effectiveness of the algorithm in treating soft inclusions with negative stiffness, a challenging issue that has not been tackled in prior art. Large amount of numerical examples are demonstrated.

In this paper, the author only presents the method in a two dimensional setting. Its generalization to three dimensional domains requires more elaborated technicalities that deserve a separate discussion.

Let Ω a bounded polygonal domain in R^2. In order to deal with soft inclusions, we let Ω_1 and Ω_2 be sub-domains of Ω such that

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset.$$

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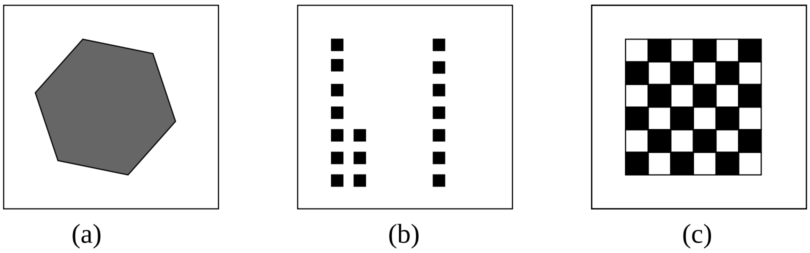

FIGURE 1. In (a) the shaded region is the unknown plastic region. In (b) and (c), the material on Ω_1 (black) is soft while the material on Ω_2 (white) is hard. The variation or oscillation in stiffness has risen to the extent that the standard algorithms are impeded in speed or accuracy.

We assume that Ω_1 is a union of small polygons that is not necessarily connected. Three examples of the domain are illustrated in Figure 1 that foreshadow the challenge we will face in the computation. In (a), the shaded area represents an unknown plastic region. In (b), the domain can be used to model material defects or random inclusions. Domain (c) is a familiar semi-periodic situation in composite material, often seen in the homogenization theory.

In order to avoid excessive technical details, we further simplify the partial differential operator to include only the principle part, given by

$$Lu = -\sum_{j=1}^{2} \left[a_j(x, \nabla u)\right]_{x_j}$$

where each a_j is a measurable function on $\Omega \times R^2$. While the dependence on the solution itself in a_j and lower order terms can also be considered, we will omit such complications. Throughout the paper, we will make the following assumptions on the coefficients of L.

(A0) For each $x \in \Omega$, $a_j(x, 0, 0) = 0$ for $j = 1, 2$.

(A1) For each $k = 1, 2$, there exists a constant $\alpha_k \geq 0$ such that for all $x \in \Omega_k$ and for all $\xi, \eta \in R^2$

$$\sum_{j=1}^{2} \left[a_j(x, \xi) - a_j(x, \eta)\right](\xi_j - \eta_j) \geq \alpha_k |\xi - \eta|^2,$$

where $|\cdot|$ denotes the Euclidean norm of R^2.

(A2) For each $k = 1, 2$, there exists a constant β_k such that for all $x \in \Omega_k$ and for all $\xi, \eta \in R^2$

$$\sum_{j=1}^{2} |a_j(x, \xi) - a_j(x, \eta)| \leq \beta_k |\xi - \eta|.$$

The sharp jumps in the stiffness coefficients are not explicitly expressed in the assumptions (A1)-(A2), but rather, embedded as a special case. In the event that

$$\alpha_1 = \delta \alpha_2, \quad \beta_1 = \delta \beta_2 \tag{1.1}$$

for a sufficiently small δ, such situations will occur. A typical range of δ can be $10^{-2} \sim 10^{-6}$ for example.