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DISCRETIZATION METHODS FOR SEMILINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. We consider an optimal control problem described by semilinear parabolic partial differential equations, with control and state constraints. Since this problem may have no classical solutions, it is also formulated in the relaxed form. The classical control problem is then discretized by using a finite element method in space and the implicit Crank-Nicolson midpoint scheme in time, while the controls are approximated by classical controls that are bilinear on pairs of blocks. We prove that strong accumulation points in L^2 of sequences of optimal (resp. admissible and extremal) discrete controls are optimal (resp. admissible and weakly extremal classical) for the continuous classical problem, and that relaxed accumulation points of sequences of optimal (resp. admissible and extremal relaxed) discrete controls are optimal (resp. admissible and weakly extremal relaxed) for the continuous relaxed problem. We then apply a penalized gradient projection method to each discrete problem, and also a progressively refining version of the discrete method to the continuous classical problem. Under appropriate assumptions, we prove that accumulation points of sequences generated by the first method are admissible and extremal for the discrete problem, and that strong classical (resp. relaxed) accumulation points of sequences of discrete controls generated by the second method are admissible and weakly extremal classical (resp. relaxed) for the continuous classical (resp. relaxed) problem. For nonconvex problems whose solutions are non-classical, we show that we can apply the above methods to the problem formulated in Gamkrelidze relaxed form. Finally, numerical examples are given.

Key Words. Optimal control, parabolic systems, discretization, piecewise bilinear controls, penalized gradient projection method, relaxed controls.

1. Introduction

We consider an optimal distributed control problem for systems governed by a semilinear parabolic boundary value problem, with control and state constraints. The problem is motivated, for example, by the control of a heat (or another, e.g. pollution) diffusion process involving a source, which is nonlinear in the heat and temperature, with a possibly nonconvex cost, resulting in an optimal control problem, which is not necessarily convex. The scope of this paper is the study of discretization/optimization methods generating classical controls (instead of relaxed ones used in our previous work, see [4]-[7] for the numerical solution of nonconvex optimal control problems (but with a convex control constraint set), which may

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have classical, or non-classical relaxed, solutions. The problem is therefore also formulated in relaxed form, using Young measures. The classical control problem is then discretized by using a Galerkin finite element method with continuous piecewise linear basis functions in space and the implicit Crank-Nicolson midpoint scheme in time, while the controls are approximated by classical controls that are bilinear on pairs of blocks. We have adopted the midpoint scheme since it gives good state approximation (under some smoothness) and yields a simple and purely symmetric matching backward scheme for the adjoint discretization. On the other hand, discontinuous double-blockwise bilinear controls generally give better overall approximation of smooth, and in some cases piecewise smooth, optimal controls, than blockwise constant ones (see numerical examples). They are well defined on pairs of blocks due to the midpoint scheme used, and for consistency with minimizations involving the Hamiltonian in the algorithms. We first state various useful necessary optimality conditions for the continuous classical and relaxed problems, and for the discrete problem. Under appropriate assumptions, we prove that strong accumulation points in L^2 of sequences of optimal (resp. admissible and extremal) discrete controls are optimal (resp. admissible and weakly extremal classical) for the continuous classical problem, and that relaxed accumulation points of sequences of optimal (resp. admissible and extremal relaxed) discrete controls are optimal (resp. admissible and weakly extremal relaxed) for the continuous relaxed problem. We then apply a penalized gradient projection method to each discrete problem, and also a corresponding discrete method to the continuous classical problem, which progressively refines the discretization during the iterations, thus reducing computing time and memory. Under appropriate assumptions, we prove that accumulation points of sequences generated by the fixed discretization method are admissible and extremal for the discrete problem, and that strong classical (resp. relaxed) accumulation points of sequences of discrete controls generated by the progressively refining method are admissible and weakly extremal classical (resp. relaxed) for the continuous classical (resp. relaxed) problem. For nonconvex problems whose solutions are non-classical, we show that we can apply the above methods to the problem formulated in Gamkrelidze relaxed form. Using a standard procedure, the computed Gamkrelidze controls can then be approximated by classical ones. For nonconvex problems with smooth (or in some cases piecewise smooth) classical solutions, the proposed discrete penalized gradient projection method often yields very accurate numerical results. On the other hand, and if the control constraint set convex, the Gamkrelidze formulation approach seems to give better results than pure relaxed methods proposed in previous work (see e.g. [3]) when dealing with nonconvex problems with non-classical solutions, since the approximation of the relaxed control by highly oscillating classical controls is replaced by the approximation of three, possibly piecewise smooth, classical ones. Finally, several numerical examples are given. For approximation of nonconvex optimal control and variational problems, and of Young measures, see [1]-[7], [10]-[12].

2. The Continuous Optimal Control Problem

Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary Γ, and let $I = (0, T)$, $T < \infty$, be an interval. Consider the semilinear parabolic state equation

$$y_t + A(t)y = f(x, t, y(x, t), w(x, t)) \text{ in } Q = \Omega \times I,$$

$$y(x, t) = 0 \text{ in } \Sigma = \Gamma \times I \text{ and } y(x, 0) = y^0(x) \text{ in } \Omega,$$