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SYMMETRIC INTERIOR PENALTY DG METHODS FOR THE COMPRESSIBLE NAVIER–STOKES EQUATIONS II: GOAL–ORIENTED A POSTERIORI ERROR ESTIMATION

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Abstract. In this article we consider the application of the generalization of the symmetric version of the interior penalty discontinuous Galerkin finite element method to the numerical approximation of the compressible Navier–Stokes equations. In particular, we consider the a posteriori error analysis and adaptive mesh design for the underlying discretization method. Indeed, by employing a duality argument (weighted) Type I a posteriori bounds are derived for the estimation of the error measured in terms of general target functionals of the solution; these error estimates involve the product of the finite element residuals with local weighting terms involving the solution of a certain dual problem that must be numerically approximated. This general approach leads to the design of economical finite element meshes specifically tailored to the computation of the target functional of interest, as well as providing efficient error estimation. Numerical experiments demonstrating the performance of the proposed approach will be presented.

Key Words. Discontinuous Galerkin methods, a posteriori error estimation, adaptivity, compressible Navier–Stokes equations

1. Introduction

In the recent series of articles [12, 13, 14, 17], we have been concerned with the development of so–called 'goal–oriented' a posteriori error estimation for h–version adaptive discontinuous Galerkin finite element methods (DGFEMs, for short) applied to inviscid compressible fluid flows; see also [19] and the references cited therein for the generalization to the hp–version of the DGFEM. Here, in contrast to traditional a posteriori error estimation which seeks to bound the error with respect to a given norm, goal–oriented a posteriori error estimation bounds the error measured in terms of certain output or target functionals of the solution of real or physical interest. Typical examples include the mean value of the field over the computational domain Ω, the normal flux through the outflow boundary of Ω, the evaluation of the solution at a given point in Ω and the drag and lift coefficients of a body immersed in a fluid. For related work, we refer to [6, 18], for example.

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The purpose of this article is to extend our earlier work on nonlinear systems of first–order hyperbolic conservation laws to the compressible Navier–Stokes equations. As in the companion article [16], the discretization of the leading order terms is performed by employing the generalization of the symmetric version of the interior penalty DGFEM. One of the key aspects of this discretization scheme is the satisfaction of the adjoint consistency condition, cf. [1], for linear problems. This condition is essential to guarantee that the optimal order of convergence of the numerical approximation to the underlying analytical solution is attained when the discretization error is measured in terms of either the L_2–norm, or in the 'goal–oriented' setting, in terms of a given target functional of practical interest. By employing a duality argument we derive a weighted, or Type I, a posteriori error bound which reflects the error creation and error propagation mechanisms inherent in viscous compressible fluid flows. On the basis of this a posteriori estimate, we design and implement the corresponding adaptive algorithm to ensure both the reliable and efficient control of the error in the prescribed target functional of interest. The superiority of the proposed approach over standard mesh refinement algorithms which employ (unweighted) empirical error indicators will be demonstrated. Additionally, we show numerically that the computed error representation formula can be employed to determine a improved value of the computed target functional $J(\cdot)$ of interest in order to yield a higher–order approximation to the exact value of this quantity.

The paper is structured as follows. After introducing, in Section 2, the compressible Navier–Stokes equations, in Section 3 we formulate its discontinuous Galerkin finite element approximation. Then, in Section 4 we derive an error representation formula together with the corresponding (weighted) Type I and (unweighted) Type II a posteriori error bounds for general target functionals of the solution. The error representation formula stems from a duality argument and includes computable residual terms multiplied by local weights involving the dual solution; the inclusion of the dual solution in the Type I bound ensures that the error creation and error propagation mechanisms inherent in viscous compressible fluid flows are reflected by the resulting local error indicators. On the basis of the (approximate) Type I error bound, in Section 5 we design and implement an adaptive algorithm that produces meshes specifically tailored to the efficient computation of the target functional of practical interest. The performance of the proposed adaptive strategy, and the quality of the (approximate) error representation formula and (approximate) Type I a posteriori bound, are then studied in Section 6 through a series of numerical experiments. Finally, in Section 7 we summarize the work presented in this paper and draw some conclusions.

The work presented in this paper is a complete and improved account of our recent work announced in the conference article [15].