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hp–VERSION INTERIOR PENALTY DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS ON ANISOTROPIC MESHES

EMMANUIL H. GEORGOULIS

Abstract. We consider the *hp*-version interior penalty discontinuous Galerkin finite element method (*hp*-DGFEM) for linear second-order elliptic reaction-diffusion-advection equations with mixed Dirichlet and Neumann boundary conditions. Our main concern is the extension of the error analysis of the *hp*-DGFEM to the case when anisotropic (shape-irregular) elements and anisotropic polynomial degrees are used. For this purpose, extensions of well known approximation theory results are derived. In particular, new error bounds for the approximation error of the L^2- and H^1-projection operators are presented, as well as generalizations of existing inverse inequalities to the anisotropic setting. Equipped with these theoretical developments, we derive general error bounds for the *hp*-DGFEM on anisotropic meshes, and anisotropic polynomial degrees. Moreover, an improved choice for the (user-defined) discontinuity-penalisation parameter of the method is proposed, which takes into account the anisotropy of the mesh. These results collapse to previously known ones when applied to problems on shape-regular elements. The theoretical findings are justified by numerical experiments, indicating that the use of anisotropic elements, together with our newly suggested choice of the discontinuity-penalisation parameter, improves the stability, the accuracy and the efficiency of the method.

Key Words. discontinuous Galerkin, finite element methods, anisotropic meshes, equations with non-negative characteristic form.

1. Introduction

In recent years, there has been an increasing interest in a class of non-conforming finite element approximations of elliptic boundary-value problems, usually referred to as *discontinuous Galerkin finite element methods*. Justifications for the renewed interest in these methods, which date back to the 1970s and the early 1980s [23, 29, 2], can be found in the attractive properties they exhibit, such as increased flexibility in mesh design (irregular grids are admissible), the freedom of choosing the elemental polynomial degrees without the need to enforce any conformity requirements, good local conservation properties of the state variable, and good stability properties near boundary/interior layers or even discontinuities [6]. The first two reasons mentioned above make discontinuous Galerkin methods very suitable contenders for *hp*-adaptivity, whereas the last two render these methods attractive when convection is the dominant feature of the problem. New error analyses for various DGFEMs have been presented in the literature during the last

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decade: see [7] for the developments until 1999 and [3] for the contemporary unified approach; see also [5].

In this work we analyze the use of anisotropic finite elements for the numerical approximation of second-order equations with non-negative characteristic form [24]. On isotropic meshes, discontinuous Galerkin finite element methods for such equations were considered in [18]. In many practical examples of boundary value problems for partial differential equations with non-negative characteristic form, diffusion can be small, degenerate, or even identically equal to zero in subregions of the computational domain. Hence, computationally demanding features may appear in their analytical solutions; these include boundary/interior layers or discontinuities in subregions where the problem becomes of first-order hyperbolic type. When structures such as layers or discontinuities are present in the solution, the use of anisotropic elements aims to provide the necessary resolution in the directions along these structures in order to reduce the number of degrees of freedom required to accurately capture them. Therefore, the combination of discontinuous Galerkin finite element methods, that produce stable approximations in the unresolved regimes, and of the use of anisotropic elements and elemental bases with anisotropic polynomial degree, that aim to provide the desired resolution only in the space directions required, is an appealing technique for the numerical solution of these problems. This work extends the arguments presented in [18] to the anisotropic setting; much of our discussion is inspired by that paper.

Anisotropic bounds for various types of FEMs have been presented in the literature, addressing mainly the question of designing structured meshes for the robust approximation of solutions to singularly perturbed boundary-value problems that admit boundary or interior layers (see, e.g., [1, 21, 26] and the references therein). Analogous results for certain DGFEMs can be found in [30, 20]. Our approach focuses on the development of general approximation-theory-tools for anisotropic elements and their subsequent application to the error analysis of the DGFEM. Potentially, the anisotropic approximation theory developed here can also be used in other applications.

The paper is structured as follows. We begin by introducing the model problem (Section 2) and the functional analytic framework used in this work (Section 3). Along with standard Sobolev spaces, we shall make use of augmented Sobolev spaces (see [12, 13] for details), as they appear to be suitable for proving *hp*-optimal error bounds for interior penalty versions of discontinuous Galerkin finite element methods. After introducing the appropriate weak formulation (from which the method will emerge) in Section 4 and the admissible finite element spaces in Section 5, the *hp*-version interior penalty discontinuous Galerkin finite element method is introduced in Section 6. Next, we present new anisotropic approximation theory results, including bounds on the projection errors of the L^2- and H^1-projection operators in various norms (Section 7). The latter will be used in the derivation of anisotropic a-priori error bounds for the *hp*-DGFEM in the energy norm (Section 8). We shall conclude with some numerical experiments indicating that the use of anisotropic elements improves the efficiency of the method, and that the analysis presented herein, yielding an new choice of the discontinuity-penalisation parameter, improves the stability, the accuracy and the efficiency of the method.

2. Model Problem

Let Ω be a bounded open (curvilinear) polygonal domain in $\mathbb{R}^2$, and let Γ_∂ signify the union of its one-dimensional open edges, which are assumed to be sufficiently