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EFFECTS OF BASIS SELECTION AND H-REFINEMENT ON ERROR ESTIMATOR RELIABILITY AND SOLUTION EFFICIENCY FOR HIGH-ORDER METHODS IN THREE SPACE DIMENSIONS

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Abstract. Designing effective high-order adaptive methods for solving stationary reaction-diffusion equations in three dimensions requires the selection of a finite element basis, *a posteriori* error estimator and refinement strategy. Estimator accuracy may depend on the basis chosen, which in turn, may lead to unreliability or inefficiency via under- or over-refinement, respectively. The basis may also have an impact on the size and condition of the matrices that arise from discretization, and thus, on algorithm effectiveness. Herein, the interaction between these three components is studied in the context of an h-refinement procedure. The effects of these choices on the robustness and efficiency of the algorithm are examined for several linear and nonlinear problems. The results demonstrate that popular choices such as the tensor-product basis or the modified Szabó-Babuška basis have significant shortcomings but that promising alternatives exist.

Key Words. *a posteriori* error estimation, adaptivity, high-order finite element basis.

1. Introduction

Adaptive finite element methods have become ubiquitous in solving systems of nonlinear partial differential equations [6]. Two advantages of these methods are reliability and efficiency. For high-order methods the choice of the finite element space and the performance of *a posteriori* error estimators are two key elements in achieving these goals. An appropriate finite element space should be as small as possible so as to minimize the number of degrees of freedom while maintaining the appropriate order. At the same time it should lead to well-conditioned Jacobian matrices. The adaptive strategy should seek to avoid over- and under-refinement. The former leads to inefficiency while the latter undermines reliability. *A posteriori* error estimates that are asymptotically exact, that is, the error estimates converge to the true error as the grid is refined, help to alleviate these problems. Additionally such estimates should be cheap to compute. The refinement strategy and finite element space, in turn, may effect the performance of the error estimator [3].

In three dimensions the tensor-product (TP) basis on hexahedral elements represents a natural choice [10, 13, 26]. Since this basis is significantly larger than

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necessary to obtain an approximation of order p other bases have been introduced called variously hierarchical or serendipity bases [10, 13, 24, 30]. One of the most popular of these bases is the hierarchical basis of Szabó and Babuška [13, 24], henceforth referred to as the SBH basis. However, the asymptotic equivalence property, critical in obtaining hierarchical *a posteriori* error estimates (cf. below and section 3) does not hold for this basis [17]. As a result in two dimensions Adjerid *et al* [2] introduced a modified-SB-hierarchical (mSBH) basis. Their basis adds the minimal number of basis functions to the SBH basis to obtain asymptotic equivalence and is easily extended to three dimensions. As the order p increases the number of possible bases between the mSBH and TP bases also increases. The examples in section 5 show that the TP and mSBH bases are unsatisfactory. In this paper the impact of choosing bases between the mSBH and TP bases on an *a posteriori* error estimator and on solution efficiency is studied.

Numerous *a posteriori* error estimation strategies have been proposed and implemented [4, 6, 25]. One widely-studied class is the hierarchical estimators [4, 25]. Among these estimators is a family of methods that rely on finding an interpolant that is asymptotically equivalent to the finite element solution in the sense of Adjerid *et al* [2]; that is, the finite element and interpolation errors converge at the same rate with the same constant in H^1. Asymptotic equivalence leads to asymptotically exact error estimates. Yu [28] and Adjerid *et al* [2] developed such an estimator for p even. Moore [19] extended their estimator to all orders $p > 1$. Many of the convergence proofs assume an idealized (e.g. uniform) grid. An extension of the estimator to adaptive grids using interpolation error estimates is proposed (cf. section 4 and Appendix B). The influence of an h-refinement strategy on estimator reliability is examined herein.

The h-refinement adaptive code is used to solve reaction-diffusion equations of the form

$$-\Delta u = F(u, \mathbf{x}), \ \mathbf{x} = (x, y, z) \in \Omega \equiv (X_0, X_1) \times (Y_0, Y_1) \times (Z_0, Z_1), \tag{1}$$

together with Dirichlet and/or Neumann boundary conditions. Throughout I assume that (1) has a unique solution of appropriate smoothness. Equation (1) is discretized using the finite element-Galerkin method with a piecewise continuous polynomial basis of degree $p > 1$. The resulting nonlinear system is solved via Newton's method. GMRES [23] with ILUT preconditioning [22] is used to solve the linear systems.

The definition of the admissible grids and rules governing h-refinement are presented in section 2. The finite element discretization is described in section 3. In section 4 the *a posteriori* error estimators are introduced along with a convergence result on h-refined grids. Computational results for three problems are given in section 5. Some conclusions are presented in section 6.

2. Grid definition and h-refinement

The grid, Δ_Ω, for Ω is obtained by recursive trisection, beginning with Ω. Thus, the grid has an octree structure with the root corresponding to Ω. The leaf vertices of the tree are called elements (unrefined elements in [27]). Each element in the grid is made up of element interiors, faces, edges and nodes. These four building blocks are henceforth referred to collectively as **components**. The level of an element in the grid is the length of the path from the root to the element. A vertex with eight subvertices is referred to as a parent vertex and the eight subvertices are its offspring or children. Eight vertices having a common parent are called siblings. The 64 vertices having a common grandparent are called cousins. A grid is said