INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 4, Pages 439–358

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Computing and Information

ON KORN'S FIRST INEQUALITY FOR QUADRILATERAL NONCONFORMING FINITE ELEMENTS OF FIRST ORDER APPROXIMATION PROPERTIES

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(Communicated by Roger Temam)

Abstract. We investigate the Korn first inequality for quadrilateral nonconforming finite elements of first order approximation properties and clarify the dependence of the constant in this inequality on the discretization parameter h. Then we use the nonconforming elements for approximating the velocity in a discretization of the Stokes equations with boundary conditions involving surface forces and, using the result on the Korn inequality, we prove error estimates which are optimal for the pressure and suboptimal for the velocity.

Key Words. nonconforming finite elements, Korn's inequality, Stokes equations, error estimates.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz–continuous boundary $\partial\Omega$ and let Γ^D be a measurable subset of $\partial\Omega$ with a positive one–dimensional measure. The Korn first inequality (cf. e.g. [16]) states that there exists a positive constant C such that

$$|\boldsymbol{v}|_{1,\Omega} \leq C \, \|\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathsf{T}}\|_{0,\Omega} \qquad \forall\, \boldsymbol{v} \in V\,, \tag{1}$$

where

$$V = \{\boldsymbol{v} \in H^1(\Omega)^2;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma^D\}\,.$$

This inequality guarantees the coerciveness of the bilinear form which is related to weak formulations of problems from linear elasticity and fluid dynamics in which forces are prescribed on a part of the boundary of the computational domain Ω.

Let $\mathcal{T}_h$ be a triangulation of Ω consisting of shape–regular elements K satisfying the usual compatibility conditions and let V_h be a finite element space build up over $\mathcal{T}_h$ and approximating the space V. Then a discrete analogue of (1) is the inequality

$$\sum_{K\in\mathcal{T}_h} |\boldsymbol{v}_h|^2_{1,K} \leq C_h \sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^{\mathsf{T}}\|^2_{0,K} \qquad \forall\, \boldsymbol{v}_h \in V_h\,. \tag{2}$$

For clarity we shall assume that C_h denotes the smallest constant for which the inequality (2) holds. In order to derive optimal convergence results, one usually needs C_h to be bounded from above by a constant C_0 independent of h. Such a constant C_0 always exists in the conforming finite element method where $V_h \subset V$

Received by the editors November 26, 2004.

2000 *Mathematics Subject Classification.* 65N12, 65N30, 65N15.

This research was supported by the Czech Grant Agency under the grant No. 201/05/0005, by the grant MSM 0021620839 and by the Otto–von–Guericke–Universität Magdeburg.

but if V_h is a nonconforming finite element space, then the dependence of C_h on h is not clear. It may even happen that, for some $\boldsymbol{v}_h \in V_h$, the right–hand side of (2) vanishes whereas the left–hand side does not and hence the inequality (2) does not hold for any constant C_h (cf. [10], [1]). Recently it was shown in [2] and [13] that $C_h \le C_0$ holds if, for any $\boldsymbol{v}_h \in V_h$ and for any edge E of $\mathcal{T}_h$ which does not lie on $\partial\Omega \setminus \Gamma^D$, the jump $[|\boldsymbol{v}_h|]_E$ of $\boldsymbol{v}_h$ across E satisfies

$$\int_E [|\boldsymbol{v}_h|]_E \, q \, \mathrm{d}\gamma = \mathbf{0} \qquad \forall\, q \in P_1(E)\,. \tag{3}$$

Unfortunately, this condition does not hold for the most nonconforming finite elements of first order approximation properties. In fact it is known that for many of these elements the inequality $C_h \le C_0$ fails but a detailed analysis of the dependence of C_h on h is not available.

In [9], the case of the nonconforming linear triangular Crouzeix–Raviart element was studied and a modification of the discrete bilinear form a_h from Section 5 was proposed which is uniformly coercive with respect to the discrete H^1 norm. This modification can be used also for the nonconforming quadrilateral finite elements considered below and leads to optimal error estimates. However, it has been observed in numerical experiments with quadrilateral elements of first order approximation properties that the standard discretization, which is simpler to implement, already shows optimal order of convergence. Up to now there is no theoretical explanation for such a behaviour. Note that also the technique of [18] for proving convergence without using (2) cannot be applied in this case since this technique uses the property (3) which is – in general – not satisfied for first order elements.

Therefore, in this paper, we will focus our attention on a theoretical support of the unexpected behaviour of nonconforming quadrilateral finite elements of first order approximation properties observed in numerical calculations. In doing this we shall concentrate on two questions:

a) characterization of the asymptotic behaviour of the constant C_h if $h \to \infty$;
b) convergence properties of discrete solutions in the case $C_h \to \infty$.

In order to cover most of the used first order nonconforming finite element spaces on quadrilaterals, we consider a class of finite element spaces V_h constructed using spaces of the type

$$\mathrm{span}\{1, \widehat{x}, \widehat{y}, \theta(\widehat{x}) - \theta(\widehat{y})\} \tag{4}$$

defined on the reference square. A precise definition of V_h will be given in Section 2. The function θ is usually an even polynomial, e.g.,

$$\theta(x) = x^2\,, \tag{5}$$

$$\theta(x) = x^2 - \frac{5}{3}\,x^4\,, \tag{6}$$

$$\theta(x) = x^2 - \frac{25}{6}\,x^4 + \frac{7}{2}\,x^6\,. \tag{7}$$

The function (5) leads to the rotated bilinear element of [17] and the functions (6), (7) were proposed in [8] in order to improve the properties of the element of [17]. In the literature one can also find nonconforming spaces which contain the above–described space as a subspace, cf. [5], [15].

In what follows we shall consider the model case

$$\Omega = (0,1)^2\,, \qquad \Gamma^D = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])\,,$$