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A PIECEWISE CONSTANT LEVEL SET FRAMEWORK

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Abstract. In this work we discuss variants of a PDE based level set method. Traditionally interfaces are represented by the zero level set of continuous level set functions. We instead use piecewise constant level set functions, and let interfaces be represented by discontinuities. Some of the properties of the standard level set function are preserved in the proposed method. Using the methods for interface problems, we need to minimize a smooth convex functional under a constraint. The level set functions are discontinuous at convergence, but the minimization functional is smooth and locally convex. We show numerical results using the methods for segmentation of digital images.

Key Words. image segmentation, image processing, PDE, variational, level set, piecewise constant level set

1. Introduction

The level set method was proposed by Osher and Sethian in [1] as a versatile tool for tracing interfaces separating a domain Ω into subdomains. Interfaces are treated as the zero level set of higher dimensional functions. Moving the interfaces can implicitly be done by evolving level set functions instead of explicitly moving the interfaces. We give a brief introduction to the level set method in §2. For a recent survey on the level set methods see [2, 3, 4, 5]. Applications of the level set method include image analysis, reservoir simulation, inverse problems, computer vision and optimal shape design [6, 7, 8, 9]. In this work, we present variants of the level set method. The primary concern for our approach is to remove the connection between the level set functions and the signed distance function and thus remove some of the computational difficulties associated with the calculation of the Eikonal equation, see §2. Another motivation is to avoid the non-differentiability associated with the Heaviside and Delta functions used in some of the level set formulations [6, 10]. This will also turn the minimization functional into a locally convex and smooth functional. The third concern of this approach is to develop fast algorithms for level set methods. Due to the fact that the functional and the constraints for this approach are rather smooth, it is possible to apply Newton types of iterations to construct fast algorithms for the proposed model. One of the variants extends the level set models proposed in [11, 12] and it is also closely related to the phase-field methods [13, 14, 15, 16]. Our framework can be used for different applications where a domain should be divided into subdomains. In this work, we concentrate on image segmentation problems.

For a given digital image $u_0 : \Omega \to \mathbb{R}$, the aim is to separate Ω into a set of subdomains Ω_i such that $\Omega = \cup_{i=1}^{n} \Omega_i$ and u_0 is nearly a constant in each Ω_i. Having determined the partition of Ω into a set of subdomains Ω_i, one can do

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further modelling on each domain independently and automatically. One general image segmentation model was proposed by Mumford and Shah in [17]. Numerical approximations are thoroughly treated in [18]. Using this model, the image u_0 is decomposed into $\Omega = \cup_i \Omega_i \cup \Gamma$, where Γ is a curve separating the different domains. Inside each Ω_i, u_0 is approximated by a smooth function. The optimal partition of Ω is found by minimizing the Mumford-Shah functional (6). This is explained in §2. Following the Mumford-Shah formulation for image segmentation, Chan and Vese [6, 10] solved the minimization problem by using level set functions. The interface Γ is traced by the level set functions. Motivated by the Chan-Vese approach, we will in this article solve the segmentation problem in a different way, i.e. by introducing a piecewise constant level set function ϕ. Instead of using the zero level of a function to represent the interface between subdomains, we let the interface be represented implicitly by the discontinuities of a set of basis functions $\psi_i(\phi)$. In order to divide Ω into subdomains Ω_i, such that $\Omega = \cup_i \Omega_i$, we use a set of functions ψ_i satisfying $\psi_i = 1$ in Ω_i and $\psi_j = 0$ in Ω_i when $j \neq i$, see Figure 1.

The rest of this article is structured as follows. In §2 we give a brief review of the traditional level set method. Our general framework and the minimization functional used for image segmentation is formulated in §3. The segmentation problem is formulated as a minimization problem with a smooth cost functional under a constraint. We are essentially minimizing the Mumford-Shah functional associated with the new level set model. In §4 and §5 we explain our two variants of the level set method for image segmentation in more detail. Both sections include algorithms and numerical results. We conclude with a brief discussion. For a more detailed treatment of the two methods, including more numerical results we refer the reader to [19, 20].

2. Standard Level Set Methods

The main idea behind the level set formulation is to represent an interface $\Gamma(t)$ bounding a possibly multiply connected region in $\mathbb{R}^n$ by a Lipschitz continuous function ϕ, having the following properties

$$\begin{cases} \phi(x,t) > 0, & \text{if x is inside } \Gamma, \\ \phi(x,t) = 0, & \text{if x is at } \Gamma, \\ \phi(x,t) < 0, & \text{if x is outside } \Gamma. \end{cases} \tag{1}$$

Some regularity must be imposed on ϕ to prevent the level set function of being too steep or too flat near the interface. This is normally done by requiring ϕ to be a signed distance function to the interface

$$\begin{cases} \phi(x,t) = d(\Gamma,x), & \text{if x is inside } \Gamma, \\ \phi(x,t) = 0, & \text{if x is at } \Gamma, \\ \phi(x,t) = -d(\Gamma,x), & \text{if x is outside } \Gamma, \end{cases} \tag{2}$$

where $d(\Gamma, x)$ denotes Euclidean distance between x and Γ. Having defined the level set function ϕ as in (2), there is a one to one correspondence between the curve Γ and the function ϕ. The distance function ϕ obeys the Eikonal equation

$$|\nabla\phi| = 1. \tag{3}$$

The solution of (3) is not unique in the distributional sense. Finding the unique vanishing viscosity solution of (3) is usually done by solving the following initial value problem to steady state

$$\phi_t + sgn(\tilde{\phi})(|\nabla\phi| - 1) = 0 \tag{4}$$

$$\phi(x,0) = \tilde{\phi}(x). \tag{5}$$