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ON GLOBAL ASYMPTOTIC STABILITY OF SOLUTIONS OF SOME IN-ARITHMETIC-MEAN-SENSE MONOTONE STOCHASTIC DIFFERENCE EQUATIONS IN $\mathbb{R}^1$

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Abstract. Global almost sure asymptotic stability of the trivial solution of some nonlinear stochastic difference equations with in-the-arithmetic-mean-sense monotone drift part and diffusive part driven by independent (but not necessarily identically distributed) random variables is proven under appropriate conditions in $\mathbb{R}^1$. This result can be used to verify asymptotic stability of stochastic-numerical methods such as partially drift-implicit trapezoidal methods for nonlinear stochastic differential equations with variable step sizes.

Key Words. Stochastic difference equations, global asymptotic stability, almost sure stability, stochastic differential equations, and partially drift-implicit numerical methods.

1. Introduction

Suppose that a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ with filtrations $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is given. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a one-dimensional real-valued $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$-martingale-difference (for details see [6]) and $\mathcal{B}(S)$ denote the set of all Borel-sets of the set S. Furthermore, let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers and $k > 0$ be a fixed integer constant.

Throughout this paper we consider discrete time stochastic difference equations (DSDEs) of the type

$$(1) \qquad x_{n+1} - x_n = -\alpha_n x_n^{2k} \left(\frac{x_{n+1} + x_n}{2} \right) + \sigma_n((x_l)_{0\le l\le n})\, \xi_{n+1},\ n \in \mathbb{N}$$

with in-the-arithmetic-mean-sense monotone drift parts

$$a_n(x_n, x_{n+1}) = -\alpha_n x_n^{2k} \left(\frac{x_{n+1} + x_n}{2} \right),$$

driven by square-integrable martingale-differences $\xi = \{\xi_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^1$ with $\mathbb{E}[\xi_n] = 0$ and $\mathbb{E}[\xi_n]^2 < +\infty$. We are especially interested in conditions ensuring the asymptotic stability of trivial solutions of these DSDEs (1). The main result should be such that it can be applied to numerical methods for related continuous time

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stochastic differential equations (CSDEs) as its potential limits. For example, consider

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t \tag{2}$$

driven by standard Wiener process $W = \{W_t\}_{t\geq 0}$ and interpreted in the Itô sense, where $a, b : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are smooth vector fields. Such CSDEs (2) can be discretized in many ways, e.g., see [20] for an overview. However, only a few of those discretization methods are appropriate to tackle the problem of almost sure asymptotic stability of its trivial solutions. One of the successful classes is that of **partially drift-implicit trapezoidal methods** with the scheme

$$x_{n+1} = x_n + \frac{1}{2}A(t_n, x_n)\,(x_{n+1} + x_n)\,\Delta_n + b(t_n, x_n)\Delta W_n \tag{3}$$

applied to equation (2), where $a(t, x) = A(t, x)x$, $\Delta_n = t_{n+1} - t_n$ and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$, for a discretization $0 = t_0 \leq t_1 \leq ... \leq t_N = T$ of the time interval $[0, T]$. These methods provide L^2-converging approximations to (2) with rate 0.5 in the worst case under appropriate conditions on a, b. For details, see [17]. Obviously, schemes (3) applied to Itô-type CSDEs

$$dX_t = -\gamma^2[X_t]^{2k+1}\,dt + b(t, X_t)\,dW_t \tag{4}$$

possess the form of (1) with $\alpha_n = \gamma^2\Delta_n$, $\sigma_n((x_l)_{0\leq l\leq n}) = b(t_n, x_n)$, $A(t, x) = -\gamma^2 x^{2k}$ and $\Delta W_n = \xi_{n+1}$. Thus, assertions on the asymptotic stability of the trivial solution of (1) help us to understand the qualitative-asymptotic behavior of methods (3) and give criteria for choosing possibly variable step sizes Δ_n for long term numerical integration such that asymptotic stability can also be guaranteed for the discretization of the related continuous time system too.

In passing we note that, that several authors have dealt with asymptotic moment-stability of stochastic-numerical methods for CSDEs. Just to name a few of them, Abukhaled and Allen [1] on expectation stability, and Artemiev [3], Artemiev and Averina [4], Mitsui and Saito [8] and Schurz [15], [16], [18], [20] with respect to mean square stability and Schurz [19] on estimates of (nonlinear) moment-stability exponents. Most of them have only treated linear equations. Moreover, very little is known on almost sure asymptotic stability for stochastic numerical methods when applied to (nonlinear) CSDEs (2). In view of equation (1), more precisely speaking, the bilinear case with $k = 0$, moment stability issues have been examined for the corresponding drift-implicit trapezoidal methods with equidistant step sizes Δ in [18]. Here we concentrate us on the nonlinear and nonautonomous subclasses of (1) exclusively, in particular, on the case with variable step sizes Δ_n.

The paper is organized in 5 sections. We suppose that the reader is familiar with basic facts on stochastic calculus, although we provide some in Section 2. This section lists some of the most important auxiliary results known from literature to prove our main result on asymptotic stability of difference equations (1) in Section 3. Section 4 discusses its applicability to the numerical approximation of stochastic differential equations illustrated by partially drift-implicit methods. Eventually, section 5 closes this paper with some final concluding remarks.

2. Auxiliary statements and Definition

The following Lemma 1 is a generalization of Doob decomposition of submartingales (for details, see [6]). Throughout the paper, we abbreviate the expression