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A PRIORI AND A POSTERIORI ERROR ESTIMATES FOR BOUSSINESQ EQUATIONS

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Abstract. This paper deals with an incompressible viscous flow problem, where the Navier-Stokes equations are coupled with a nonlinear heat equation. Existence and uniqueness results are established. Next, a finite element approximation of the problem is presented and analyzed. Error estimates are obtained and a posteriori error estimate is given.

Key Words. Boussinesq equations, a posteriori error estimates, finite element methods.

1. Introduction

In this paper, we are interested in an incompressible viscous fluid governed by Navier-Stokes equations, when they are coupled with a nonlinear heat equation by the intermediary of the reaction source term. The considered model is the system formed by the equations describing the flow, under the approximation of *Boussinesq*. Within the framework of this approximation, we do not take account of the variation of density. Therefore the density is regarded as constant in the equation of mass conservation. The *Boussinesq* approximation was justified and used to study some chemical phenomena as in [10, 11]. Numerical analysis and finite element approximation of this model, in non stationary form, is studied in [1, 9]. In this work, we are interested in a similar model, but in a stationary form.

Let Ω an open bounded convex domain of $\mathbb{R}^d$ (d=2,3), with Lipschitz continuous boundary Γ . In Ω, we consider the following stationary model:

$$
(P)\begin{cases}
-\Delta T + u.\nabla T + f(T) = 0, \quad \text{in } \Omega, \\[2mm]
-\mu\Delta u + (u.\nabla)u + \nabla p = \ F(T), \quad \text{in } \Omega, \\[2mm]
div\ u = 0, \\[2mm]
u = 0 \ \text{ and } T = 0, \quad \text{on } \Gamma,
\end{cases}
$$

where the unknown factors are speed u, the pressure p and the temperature T; the coefficient μ (the viscosity of the fluid) is assumed to be positive. The data are a regular function F of $\mathbb{R}$ to $\mathbb{R}^d$ (typically, the function F is a gravity force proportional to the variations of density, therefore dependents on the temperature) and an other regular function f of $\mathbb{R}$ to $\mathbb{R}_+^*$ (typically, the function f is the source term of the reaction depending on the temperature and also on energy; usually this

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function is obtained by the *Arrhenius* law). On datas, we assume that the first and the second derivatives are bounded.

This model has been studied by using topological degree theory to prove the existence results in [2] and by using mixed-dual variational formulation in two dimensions in [6, 7], the authors of these last works introduced the gradient of velocity and the gradient of temperature as unknowns, on which, they give some a priori error estimates.

In the next section, we prove a result of existence and uniqueness of the continuous problem. In the third section, Some usual finite element spaces are introduced, for speed, for the pressure and for the temperature. A discrete problem is given, we prove some error estimates on the speed, on the pressure and on the temperature. Finally in the last section, a posteriori error estimate is given.

2. Existence and uniqueness

The variational form of the problem (P) can be written as following:

$$(P0)\begin{cases} \text{Find } (u,p,T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega) \text{ such that} \\ \forall v \in (H_0^1(\Omega))^d, \mu \int_\Omega \nabla u.\nabla v dx + \int_\Omega [(u.\nabla)u] v dx - \int_\Omega p div\ v dx \\ \qquad\qquad\qquad\qquad = \int_\Omega F(T) v dx, \\ \forall q \in L_0^2(\Omega), \quad \int_\Omega q div\ u dx = 0, \\ \forall s \in H_0^1(\Omega), \int_\Omega (\nabla s \nabla T + u.\nabla T) dx + \int_\Omega f(T) s dx = 0. \end{cases}$$

First of all, we will rewrite the problem in an equivalent form, allowing us to prove the existence of the weak solution. For that, we introduce the spaces:

$$V = \{v \in (H_0^1(\Omega))^d, \quad div = 0\} \quad \text{and} \quad Y = V \times H_0^1(\Omega).$$

Let $A(.,.)$ the map defined by:

$$\forall((u,T),(v,s)) \in Y^2,$$

$$A((u,T),(v,s)) = \int_\Omega (\mu \nabla u \nabla v + (u.\nabla)uv) dx - \int_\Omega F(T) v dx$$
$$+ \int_\Omega \nabla T.\nabla s dx + \int_\Omega (u.\nabla T) s dx + \int_\Omega f(T) s dx.$$

We consider the problem

$$(P1)\begin{cases} \text{Find}(u,T) \in V \times H_0^1(\Omega), \text{such that} \\ \forall (v,s) \in V \times H_0^1(\Omega), \quad A((u,T),(v,s)) = 0. \end{cases}$$

It is easy to see that, if the triplet $(u,p,T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0^1(\Omega))$ is solution of $(P0)$, then (u,T) is solution of $(P1)$. Reciprocally, for any solution $(u,T) \in V \times H_0^1(\Omega)$ of $(P1)$, there exist a unique element p of $L_0^2(\Omega)$ such that the triplet (u,p,T) is solution of $(P0)$. To prove the existence of the solution for the problem $(P1)$, we need the following theorem: