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A VARIABLE PRECONDITIONING USING THE SOR METHOD FOR GCR-LIKE METHODS

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Abstract. We propose a variant of variable preconditioning for Generalized Conjugate Residual (GCR)-like methods. The preconditioning is carried out by roughly solving $A\boldsymbol{z} = \boldsymbol{v}$ by an iterative method to a certain degree of accuracy instead of computing $K\boldsymbol{z} = \boldsymbol{v}$ in a conventional preconditioned algorithm. In our proposal, the number of iterations required for computing $A\boldsymbol{z} = \boldsymbol{v}$ is changed at each iteration by establishing a stopping criterion. This enables the use of a stationary iterative method when applying different preconditioners. The proposed procedure is incorporated into GCR, and the mathematical convergence is proved. In numerical experiments, we employ the Successive Over-Relaxation (SOR) method for computing $A\boldsymbol{z} = \boldsymbol{v}$, and we demonstrate that GCR with the variable preconditioning using SOR is faster and more robust than GCR with an incomplete LU preconditioning, and the FGMRES and GMRESR methods with the variable preconditioning using the Generalized Minimal Residual (GMRES) method. Moreover, we confirm that different preconditioners are applied at each iteration.

Key Words. Linear systems, generalized conjugate residual method, generalized minimal residual method, variable preconditioning, inner-loop and outer-loop.

1. Introduction

Let us consider a preconditioning for the Krylov subspace (KS) method for solving a large sparse system

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where A is a nonsingular $n \times n$ matrix, and the right-hand side vector $\boldsymbol{b}$ is an n-vector.

A preconditioning strategy is a means to enhance the convergence by transforming the original system (1). The preconditioning is performed as follows: First, we construct a preconditioner K that approximates the coefficient matrix A under the assumption that $K^{-1}\boldsymbol{v}$ can be solved more easily and faster than computing $A^{-1}\boldsymbol{v}$, where the computation of $K^{-1}\boldsymbol{v}$ is involved in a conventional preconditioned KS algorithm. An incomplete LU (ILU) factorization ([4, 12]) is frequently used for constructing the preconditioner K. Next, the linear system $K\boldsymbol{z} = \boldsymbol{v}$ is computed for $\boldsymbol{z}$ by a direct method at each iteration of the preconditioned algorithm.

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The FGMRES ([14, 15]) and GMRESR ([18]) methods have recently been proposed as variants of the Generalized Minimal Residual (GMRES) method ([13]). The characteristics for these methods are that different preconditioners can be applied at each iteration. So the preconditioning is referred to as *variable preconditioning*. This is a new framework of preconditioning. The preconditioning in FGMRES and GMRESR is performed by obtaining an approximation to $A^{-1}\boldsymbol{v}$, i.e., roughly solving $A\boldsymbol{z} = \boldsymbol{v}$. A KS method based on a minimum residual approach like GMRES is then employed for computing $A\boldsymbol{z} = \boldsymbol{v}$, and the number of iterations required for calculating $A\boldsymbol{z} = \boldsymbol{v}$ is determined so that the number is same at each iteration. Moreover, several authors ([2, 9, 16]) have worked on the idea of applying different preconditioners at each iteration and proposed variants of the Conjugate Gradient (CG) ([10]), Generalized Conjugate Gradient (GCG) ([1]) and Quasi-Minimal Residual (QMR) ([7]) methods. [21] also demonstrates that GMRESR is effective for practical applications.

In contrast, we propose a variant of the preconditioning in which different preconditioners can be applied at each iteration. The basic idea is to obtain an approximation to $A^{-1}\boldsymbol{v}$ instead of computing $K^{-1}\boldsymbol{v}$. That is to say, the expression $A\boldsymbol{z} = \boldsymbol{v}$ is roughly solved by an iterative method to a certain degree of accuracy. In our proposal, the iteration for computing $A\boldsymbol{z} = \boldsymbol{v}$ is stopped according to the accuracy of approximation and the maximum number of iterations. As a result, the number of iterations can be changed at each iteration. This enables the use of a stationary iterative (SI) method, such as the Successive Over-Relaxation (SOR) method ([8, 19]), when applying different preconditioners. On the other hand, a KS method must be used to enable different preconditioners to be applied in FGMRES and GMRESR since the stopping criterion to change the number of iterations is not provided. Consequently, the convergence behavior of the Generalized Conjugate Residual (GCR)-like ([5]) methods with our procedure is different from that of FGMRES and GMRESR.

This paper is organized as follows. In §2, the basic idea of the proposed preconditioning is described. The GCR algorithm with the variable preconditioning is presented, and the convergence rate for the algorithm is also given. Moreover, a suitable method and stopping criterion for computing $A\boldsymbol{z} = \boldsymbol{v}$ are discussed on the basis of the theorem. Finally, the differences between GCR with our proposed procedure, FGMRES and GMRESR with the original variable preconditioning are summarized. In section 3, through numerical experiments we demonstrate that GCR with the variable preconditioning using SOR is faster and more robust than GCR with the ILU(0) preconditioning (abbreviated as ILU(0)-GCR), ILU(1)-GCR, and FGMRES and GMRESR using GMRES with the ILU(0) preconditioning (abbreviated as ILU(0)-GMRES). Moreover, we confirm that different preconditioners are applied at each iteration. Conclusions are given in §4.

2. A variant of variable preconditioning

2.1. Variable preconditioning. This subsection describes the basic idea of our proposed variable preconditioning and explains how the preconditioning is incorporated into GCR.