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THE INTERPOLATED COEFFICIENT FEM AND ITS APPLICATION IN COMPUTING THE MULTIPLE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. Convergence and superconvergence of the interpolated coefficient finite element method (ICFEM) are discussed as the ICFEM reduces the computation cost greatly. Further, the ICFEM is implemented to compute the multiple solutions of some semilinear elliptic problems.

Key Words. Superconvergence, convergence, the interpolated coefficient finite element method, semilinear elliptic problem, multiple solutions.

1. Introduction

As semilinear partial differential equations arise in physics, biology, energy, and engineering, their study has attracted the attention of many pure and applied mathematicians and physicists. It is well known that the standard finite element method plays a very important role in solving these problems. Unfortunately, the computation cost for implementing the finite element method is usually very expensive.

To overcome this difficulty, a simple and graceful idea called the interpolated coefficient finite element method (ICFEM), which was originally inspired by solving semilinear parabolic problems, was proposed by M. Zlámal [12] et al. Further, he obtained the error estimate $\|(u_h - u)(t)\| = O(h^2)$ for the linear element solution $u_h(t)$ with an unproven assumption that $||u_h(t)||_\infty$ is bounded. Later, Larsson, Thomée, and Zhang [9] studied the linear triangular finite element solution $u_h(t)$ and obtained the error estimate $\|(u_h - u)(t)\| = O(h)$. In [3], implementing some superconvergence techniques, Chen, Larsson, and Zhang derived an almost optimal convergence order $\|(u_h - u)(t)\| = O(h^2 \ln h)$ on piecewise uniform triangular meshes.

In this paper, we show that the interpolated coefficient finite element method for solving the semilinear elliptic equations has the same convergence order or even superconvergence properties as those of the standard finite element method. Moreover, combined with the Improved Search-extension Method [4, 11], the ICFEM is used to compute the multiple solutions of some typical semilinear elliptic problems.

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2. Convergence and superconvergence of the ICFEM

For completeness, below the interpolated coefficient finite element method for solving semilinear elliptic problems is introduced first.

Consider a semilinear elliptic problem with zero Dirichlet boundary condition, i.e.,

$$-D_i(a_{ij}D_ju) + au + f(u) = 0 \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega, \tag{1}$$

with its weak form

$$Q(u, v) = A(u, v) + (f(u), v) = 0, \; \forall v \in S_0, \tag{2}$$

where Ω is a 2-dimensional bounded domain with Lipschitz boundary $\partial\Omega$, $S_0 = \{u \in H^1(\Omega), u = 0 \text{ on } \partial\Omega\}$, and the bilinear form

$$A(u, v) = \int_\Omega (a_{ij}(x)D_iuD_jv + a(x)uv)dx$$

is assumed to be bounded and S_0-coercive.

We assume that the domain Ω is subdivided into a finite number of elements τ with the subdivision J^h and let $Z_h = \{x_j\}_1^M$ be the set of all interior nodes. Denote by $S^h \subset S_0$ the n-degree finite element subspace and $\{N_j(x)\}_1^M$ the bases of S^h. It is well known that the standard finite element solution $u_h \in S^h$ of (1) can be expressed as $u_h(x) = \sum\limits_{j=1}^{M} U_jN_j(x) \in S^h$, $U_j = u_h(x_j)$, and satisfies

$$A(u_h, v) + (f(u_h), v) = 0, \forall v \in S^h. \tag{3}$$

By taking $v = N_i$, $i = 1, 2, ..., M$, (3) leads to a nonlinear system of equations

$$\sum_{j=1}^{M} A(N_j, N_i)U_j - (f(\sum_{j=1}^{M} N_j(x)U_j), N_i) = 0, \;\; i = 1, 2, ...M, \tag{4}$$

which is often solved by the Newton method. It is known that the Jacobi matrix is the main concern in the implementation of the Newton method. A direct computation shows that the Jacobi matrix of (4) is

$$J = \{A(N_j, N_i) - (f'(\sum_{k=1}^{M} N_kU_k)N_j, N_i)\}_{M\times M}, \tag{5}$$

which has to be updated repeatedly as the iterations proceed. Obviously, the integrations for the second term in (5) are quite large and will result in the very time-consuming and expensive computation of the Newton method.

Now we introduce the interpolated coefficient finite element method for solving (1). Substitute the interpolation $I_hf(u_h) = \sum\limits_{j=1}^{M} N_j(x)f(U_j)$ with $U_j = u_h(x_j)$ rather than $f(u_h)$ into (3) and still denote the interpolated coefficient finite element solution $u_h = \sum\limits_{j=1}^{N} U_jN_j(x)$. Then we obtain a new finite element equation

$$A(u_h, v) + (I_hf(u_h), v) = 0, \;\; \forall v \in S^h. \tag{6}$$

As a result, we obtain a nonlinear algebraic system of equations

$$\sum_{j=1}^{M} (k_{ij}U_j + m_{ij}f(U_j)) = 0, \;\; i = 1, 2, ..., M, \tag{7}$$