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# A SUPERCONVERGENT FINITE ELEMENT SCHEME FOR THE REISSNER-MINDLIN PLATE BY PROJECTION METHODS

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**Abstract.** The Reissner-Mindlin model is frequently used by engineers for plates and shells of small to moderate thickness. This model is well known for its "locking" phenomenon so that many numerical approximations behave poorly when the thickness parameter tends to zero. Following the formulation derived by Brezzi and Fortin, we construct a new finite element scheme for the Reissner-Mindlin model using $L^2$ projections onto appropriately-chosen finite element spaces. A superconvergence result is established for the new finite element solutions by using the $L^2$ projections. The superconvergence is based on some regularity assumption for the Reissner-Mindlin model and is applicable to any stable finite element methods with regular but non-uniform finite element partitions.

**Key Words.** finite element methods, superconvergence, the method of least-squares fitting, Reissner-Mindlin plate.

## 1. Introduction

The Reissner-Mindlin plate is a mathematical model that is frequently used by engineers for plates and shells of small to moderate thickness. To describe the model, we consider a plate or a shell of thickness $t > 0$. Let $\Omega$ be the region occupied by the plate. Denote by $w = w(x, y)$ and $\boldsymbol{\phi} = (\phi_1, \phi_2)^t$ the transverse deflection of $\Omega$ and the rotation of the fibers normal to $\Omega$, respectively. The Reissner-Mindlin plate model determines $w$ and $\boldsymbol{\phi}$ as the unique solution to the following variational problem: find $(w, \boldsymbol{\phi}) \in H_0^1(\Omega)\times[H_0^1(\Omega)]^2$ such that for all $(v, \boldsymbol{\psi}) \in H_0^1(\Omega)\times[H_0^1(\Omega)]^2$

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \nabla w, \boldsymbol{\psi} - \nabla v) = (g, v), \tag{1.1}$$

where $g$ is the scaled transverse loading function, $\lambda = Ek/2(1 + \nu)$ is the shear modulus with $E$ the Young's modulus, $\nu$ the Poisson ratio, $k$ the shear correction factor. The symbol $\nabla$ denotes the gradient operator. $H^1(\Omega)$ is the Sobolev space defined by

$$H^1(\Omega) = \left\{v : \quad v \in L^2(\Omega), \nabla v \in [L^2(\Omega)]^2\right\}.$$

Here $L^2(\Omega)$ is the set of square integrable functions over the domain $\Omega$ with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions with vanishing boundary value. The bilinear form $a(\cdot, \cdot)$ in (1.1) is given by

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega [(1-\nu)\boldsymbol{\epsilon}(\boldsymbol{\phi}) : \boldsymbol{\epsilon}(\boldsymbol{\psi}) + \nu\nabla\cdot\boldsymbol{\phi}\nabla\cdot\boldsymbol{\psi}],$$

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where $\nabla\cdot$ is the divergence operator, $\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \frac{1}{2}[\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\phi}^t]$, and $-1 < \nu < \frac{1}{2}$.

An obvious numerical procedure for the Reissner-Mindlin model would be a Galerkin finite element method based on the weak formulation (1.1) in which $w$ and $\boldsymbol{\phi}$ are both approximated by continuous piecewise polynomials over a prescribed finite element partition $\mathcal{T}^h$ of $\Omega$. However, such schemes are known to have "locking" difficulty in that the resulting numerical approximations behave poorly when the thickness parameter $t$ tends to zero. Many researchers have been working on the Reissner-Mindlin model by aiming at designing efficient and "locking free" numerical schemes. Among a few of successes, we mention the work of Brezzi and Fortin [3] who derived a formulation for the Reissner-Mindlin model by introducing two variables (the irrotational and solenoidal parts of the transverse shear strain) in addition to the primitive variables (the transverse displacement and the rotation vector) and developed a finite element method which is locking free. Inspired by the work of Brezzi and Fortin, Arnold and Falk [1] developed an efficient triangular element for the Reissner-Mindlin model in the primitive variables using the $P1$ nonconforming linear element for the transverse displacement and conforming linear element with bubbles for the rotation to the Reissner-Mindlin model. They proved that the method converges with an optimal order uniformly with respect to the thickness. For more literature, the reader is referred to [5] [6], [2], [4], [9] and references therein.

The objective of this paper is to propose and analyze a modified scheme for the Brezzi-Fortin method [3], which will yield numerical approximations for the Reissner-Mindlin plate model with high order of accuracy. There are two challenges to this task: (1) modification of the Brezzi-Fortin's method, and (2) tedious analysis for the postprocessing projection method of Wang [11]. Our result has potential impact in practical computation for Reissner-Mindlin model in that it can provide an efficient a posteriori error estimator for adaptive grid local refinement.

## 2. The Brezzi-Fortin Formulation and Approximation

We first introduce some standard notations. Denote by $H^m(\Omega)$ for any integer $m \geq 0$ the Sobolev space:

$$H^m(\Omega) = \left\{v : \ \partial_x^{\alpha_1}\partial_y^{\alpha_2} v \in L^2(\Omega), \alpha_i \geq 0, \alpha_1 + \alpha_2 \leq m\right\}$$

with norm given by

$$\|v\|_s = \left(\sum_{\alpha_1+\alpha_2\leq m} \|\partial_x^{\alpha_1}\partial_y^{\alpha_2} v\|^2\right)^{\frac{1}{2}}.$$

For non-integer values of $m$, $H^m(\Omega)$ is defined via the standard interpolation method. Let $\mathcal{D}(\Omega)$ be the linear space of infinitely differentiable functions with compact support on $\Omega$. As usual, $H_0^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_s$. For any function $\boldsymbol{\phi} \in H_0^1(\Omega)$, denote its curl by

$$\nabla\times\boldsymbol{\phi} = \partial_2\phi_1 - \partial_1\phi_2.$$

Denote by $\nabla^\perp$ the formal adjoint of $\nabla\times$ given by

$$\nabla^\perp p = \begin{pmatrix} -\partial_2 p \\ \partial_1 p \end{pmatrix}, \quad p \in H^1(\Omega).$$

Following [1], without loss of generality we may assume that $\lambda = 1$ and

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = (\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\psi}).$$