INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 3, Pages 463–498

A PARTITIONED METHOD WITH DIFFERENT TIME STEPS FOR COUPLED STOKES AND DARCY FLOWS WITH TRANSPORT

JINGYUAN ZHANG, HONGXING RUI, AND YANZHAO CAO

Abstract. A decoupled finite element algorithm with different time steps on different physical variables for a Stokes-Darcy interface system coupled with the solution transport is studied. The viscosity of the Stokes equation is assumed to depend on the concentration of the transported solution. The numerical algorithm consists of two steps. In the first step, the system is decoupled on the interface. In the second step, the time derivatives are discretized with different step sizes for different partial differential equations in the system. An careful error analysis provides a guidance on the ratio of the step sizes with respect to the ratio of the physical parameters. Numerical examples are presented to verify the theoretical results and illustrate the effectiveness of the decoupled algorithm of using different time steps.

Key words. Coupled Stokes and Darcy flows, solute transport, decoupled algorithm, different time steps, error estimates.

1. Introduction

Recently there have been growing interests in building suitable mathematical and numerical models for the coupling of fluid flows in a porous medium domain and a free flow domain. In the porous medium domain, the fluid flow can be modeled by a Darcy equation while in the free flow domain the fluid flow can be modeled by a Stokes equation. The Darcy equation and the Stokes equation are coupled through conditions on the interface which connects the porous domain and the free flow domain. Modeling through the Stokes-Darcy system has a wide arrange of applications such as hydrology[5], environment science[12], and biofluid dynamics[15].

A number of numerical methods have been developed for the coupled Stokes-Darcy flow system, including the domain decomposition method [2, 39, 9], the mixed finite element method [1, 23, 38], the non-conforming finite element methods [33], the Mortar multiscale finite element methods [20], the Lagrange multiplier and mixed element methods [4, 24, 27, 16, 17], the mixed finite element method combining with the DG method [31, 32], the DG method combining with mimetic finite difference method [25], the pseudospectral least squares method [21] and spectral method [41], and many other numerical methods [18, 10, 26, 29].

The aim of this paper is to construct an efficient numerical algorithm for the Stokes-Darcy flow system coupled with an advection-diffusion equation that models, for example, the transport of a chemical. In [8], Cesmelioglu and Riviere study the existence and stability of the weak solution with the fluid viscosity depending on the concentration for this model. For numerical methods, in [40] the flow equations are solved through the domain decomposition method using classical finite element methods in the Stokes region and mixed finite element methods in the Darcy region, and the transport equation is solved by a local discontinuous Galerkin (LDG) method, while in [34] the authors proposed a mixed weak formulation and use the

Received by the editors May 19, 2018 and, in revised form, July 10, 2018.
2000 *Mathematics Subject Classification.* 65M12; 65M15; 65M60.

nonconforming piecewise Crouzeix- Raviart finite element, piecewise constant and conforming piecewise linear finite element to approximate velocity, pressure and concentration, respectively.

In this paper, we study the finite element approximation of the Stokes-Darcy-Transport system with different time steps on different physical variables. As a multi-phyiscs problem, each of the equations in the the Stokes-Darcy-Transport system has a different time scale reflected by the corresponding partial differential equation and the related physical parameters. Thus it is natural to use larger time step in the region with slower velocity. The multiple-time-step technique for the non-stationary Stokes-Darcy model was presented in [36, 37]. There the viscosity of the free flow, the hydraulic coefficient of the flow in porous medium and the diffusion coefficient of the transport are assumed to be constants. In this study, we assume that viscosity and the hydraulic conductivity depend on the concentration of the transport and the diffusion coefficient depends on the velocity of flow in the porous medium. Under a modest time step restriction in relation to physical parameters, we obtain the stability of the method and a priori error estimates. With the help of such error analysis we derive criteria of choosing the time step for each physical variables in accordance to the ratios of the physical parameters. In particular, we show that the ratio between the time steps should be proportional to the ratios between these physical parameters. For spatial discretization, we adopt the decoupling method of [28] (see also [6, 22, 37]).

The rest of the article is organized as follows. In Section 2, we introduce the model problem, and present the mixed weak formulation. Coupling and decoupling schemes, and the stability of the decoupling scheme with different time steps on different subdomains are given in Section 3. The error estimates for fluid velocity, kinematic pressure, piezometric head and concentration are presented in Section 4. Finally in Section 5, we present some numerical examples to demonstrate our theoretical results.

Through out this paper we use K and C, with or without subscription, to denote a generic constant, which may have different values in different appearances.

2. Model problem and weak formulation

The model under consideration is a flow in a bounded domain $\Omega \subset R^N$ (N=2 or 3), consisting of an fluid flow region Ω_f, where the flow is governed by the Stokes equation, and a porous medium region $\Omega_p = \Omega \backslash \overline{\Omega}_f$, where the flow is governed by the Darcy's law. Here $\Omega_l (l = f, p)$ are bounded domains with outward unit normal vectors $\boldsymbol{n}_l (l = f, p)$. The whole domain Ω is occupied by a mixture of two miscible fluids which before mixing are each incompressible, therefore the Boussinesq approximation is valid and the mean-volume velocity is thought to be equal to the mean mass velocity. The two regions are separated by an interface $\Gamma_I = \partial\Omega_f \cap \partial\Omega_p$, and $\tau_j, j = 1, \cdots, N-1$ denote an orthonormal system of tangent vector on Γ_I. On the interface, we have $\boldsymbol{n}_f = -\boldsymbol{n}_p$. Let $\Gamma_l = \partial\Omega_l \backslash \Gamma_I (l = f, p)$. Each interface and boundary is assumed to be polygonal or polyhedral. Figure 1 gives a schematic representation of the geometry with N=2.

The equations of motion, continuity and mass transport for the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$, kinematic pressure $p(\boldsymbol{x}, t)$ and concentration $c(\boldsymbol{x}, t)$ in Ω_f can be written as

$$\partial_t \boldsymbol{u} - \nabla \cdot (2\mu(c)\boldsymbol{S}(\boldsymbol{u})) + \nabla p = \boldsymbol{f}(c), \quad \boldsymbol{x} \in \Omega_f, t \in J, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \boldsymbol{x} \in \Omega_f, t \in J, \tag{2}$$