INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 3, Pages 412–435

ANALYSIS OF POLLUTION-FREE APPROACHES FOR MULTI-DIMENSIONAL HELMHOLTZ EQUATIONS

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Abstract. Motivated by our recent work about pollution-free difference schemes for solving Helmholtz equation with high wave numbers, this paper presents an analysis of error estimate for the numerical solution on the annulus and hollow sphere domains. By applying the weighted-test-function method and defining two special interpolation operators, we first derive the existence, uniqueness, stability and the pollution-free error estimate for the one-dimensional problems generated from a method based on separation of variables. Utilizing the spherical harmonics and approximations results, we then prove the pollution-free error estimate in L^2-norm for multi-dimensional Helmholtz problems.

Key words. Helmholtz equation, error estimate, finite difference method, polar and spherical coordinates, pollution-free scheme.

1. Introduction

This paper is focused on the Helmholtz equation defined as follows:

$$-\Delta\tilde{u} - k^2\tilde{u} = 0, \text{ in } \mathbb{R}^d\backslash\mathbb{B}_1, \tag{1}$$

$$(\partial_r\tilde{u} + jk\tilde{u})|_{\partial\mathbb{B}_1} = \tilde{g}_1, \tag{2}$$

$$\partial_r\tilde{u} - jk\tilde{u} = o\Big(||\mathbf{x}||^{\frac{1-d}{2}}\Big), \text{ as } ||\mathbf{x}|| \to \infty, \tag{3}$$

where k is the wave number, $\mathbb{B}_1$ is a bounded domain in $\mathbb{R}^d$, $\mathbf{x} = (x_1, \cdots, x_d)$ $(d = 1,\ 2,\ 3)$, $\tilde{g}_1$ is a given function, ∂_r denotes the radial derivative and $j^2 = -1$. Applying an absorbing boundary condition method, or the perfectly matched layers (PML) method, the problem (1)-(3) may be reduced to the following equation (see [7, 15, 16, 23, 26, 43, 44, 57]):

$$-\Delta\tilde{u} - k^2\tilde{u} = 0, \text{ in } \Omega := \mathbb{B}_2\backslash\mathbb{B}_1, \tag{4}$$

$$(\partial_r\tilde{u} + jk\tilde{u})|_{\partial\mathbb{B}_1} = \tilde{g}_1, \tag{5}$$

$$(\partial_r\tilde{u} - jk\tilde{u})|_{\partial\mathbb{B}_2} = \tilde{g}_2, \tag{6}$$

where $\mathbb{B}_2 \in \mathbb{R}^d$ $(d = 1, 2, 3)$ is a sufficiently large ball containing $\mathbb{B}_1$ and $\tilde{g}_2$ is a given function.

It is well-known that solving the Helmholtz equation with high wave numbers numerically is very difficult and challenging due to the high oscillation solutions. Moreover, the resulted linear system is indefinite and ill-conditioned (see [1, 12, 13, 14, 22, 28, 29, 30, 31, 32, 33, 34, 50]). Another difficulty is that the "pollution effect" exists in almost all computational schemes applied to multi-dimensional Helmholtz equation such that the accuracy of the numerical solution becomes totally unacceptable for the cases with high wave numbers unless very fine meshes are used in the computation. In the past several decades, many studies have been reported to eliminate or reduce the "pollution effect". For example,

Received by the editors September 1, 2018.
2000 *Mathematics Subject Classification.* 65N06, 65N15, 65N22.
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Babuška et al. [4, 47] considered the generalized finite element method to minimize the "pollution effect". Another popular technique is based on the h-p finite element method (see [19, 32, 33, 54]), in which the "pollution effect" is reduced by increasing the order of the polynomial basis function or decreasing the mesh size h. For the finite difference methods, many higher order compact schemes were developed [21, 42, 45, 46, 49]. Recently, Chen et al. [15, 16] proposed two methods, in which the numerical dispersion is minimized by choosing optimal parameters. Other computational techniques based on the spectral methods were investigated, and the reader is referred to [6, 8, 17, 20, 27, 37, 39, 43, 44, 59]. However, it is important to note that although pollution-free numerical schemes have been reported in [24, 52, 53], there does not exist any analysis results about pollution-free methods for solving the multi-dimensional problems.

To ensure the bound of the relative error for the numerical solution of the problem (4)-(6), it is usually necessary to impose the following condition

$$k^{\beta}(kh)^{\gamma} = constant. \tag{7}$$

Here, h denotes the mesh size, and two constants $\beta > 0, \gamma > 0$ are real numbers. For example, β =2, and γ =2 and 4, when the solution is computed by the standard central finite difference scheme and the compact fourth order difference scheme, respectively. Considering that the Helmholtz problem is numerically solved with a fixed value of kh, and due to the relation given in (7), the numerical error will not decrease even when the mesh size is reduced. This adverse behaviour is the direct consequence of the "pollution effect", and more detailed discussion is reported in [31]. It has been cited by Babuška and Sauter [5] that the "pollution effect" can not be avoided on a general bounded domain for the finite element approximation of two- ($2D$) and three-dimensional ($3D$) Helmholtz equations.

In this study, we focus on the pollution-free difference method. It should be noted that the standard finite difference and the higher order compact methods are constructed based on a truncated Taylor series expansion, and the truncation errors depend on the wave numbers and thus causing the "pollution effect" unavoidable. To eliminate the pollution, pollution-free difference schemes for the one-dimensional Helmholtz equation have been proposed in [25, 36, 50, 53, 56], in which the derivation takes account of all terms in the Taylor series expansion. Compared with the standard finite difference methods, the numerical error of the pollution-free scheme depends only on the mesh size h but independent of the wave number k. Therefore, the numerical error is decreasing as the mesh size is reduced [50, 56]. Consequently, the condition (7) can be relaxed to the common "rule of thumb" (i.e., 8 to 10 discrete mesh points for each wavelength), that is

$$kh = \mathcal{C}_1 \leq \frac{\pi}{4}. \tag{8}$$

Compared to (7), relatively large value of kh could be employed even when the wave number is very high. Numerical simulations reported in [52] also verify that the pollution-free difference scheme can produce a stable numerical solution even when $kh > 1$. According to the condition (7), the mesh size of the standard finite difference or compact difference schemes must satisfy the condition

$$kh \ll 1, \tag{9}$$

for problems with high wave numbers. Therefore, a pollution-free scheme is much more efficient than the standard and compact finite difference schemes. A detailed development for $1D$ problems has been reported in [50, 56]. However, this approach can not be extended directly to problems on a general domain in $2D$ and $3D$.