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GLOBAL STABILITY OF CRITICAL TRAVELING WAVES WITH OSCILLATIONS FOR TIME-DELAYED REACTION-DIFFUSION EQUATIONS

MING MEI, KAIJUN ZHANG*, AND QIFENG ZHANG

Abstract. For a class of non-monotone reaction-diffusion equations with time-delay, the large time-delay usually causes the traveling waves to be oscillatory. In this paper, we are interested in the global stability of these oscillatory traveling waves, in particular, the challenging case of the critical traveling waves with oscillations. We prove that, the critical oscillatory traveling waves are globally stable with the algebraic convergence rate $t^{-1/2}$, and the non-critical traveling waves are globally stable with the exponential convergence rate $t^{-1/2}e^{-\mu t}$ for some positive constant μ, where the initial perturbations around the oscillatory traveling wave in a weighted Sobolev can be arbitrarily large. The approach adopted is the technical weighted energy method with some new development in establishing the boundedness estimate of the oscillating solutions, which, with the help of optimal decay estimates by deriving the fundamental solutions for the linearized equations, can allow us to prove the global stability and to obtain the optimal convergence rates. Finally, numerical simulations in different cases are carried out, which further confirm our theoretical stability for oscillatory traveling waves, where the initial perturbations can be large.

Key words. Nicholson's blowflies equation, time-delayed reaction-diffusion equation, critical traveling waves, oscillation, stability, numerical simulations.

1. Introduction and main result

This is a continuation of the previous studies [4,25] on the stability of oscillatory traveling waves for a class of non-monotone reaction-diffusion equations with time-delay

$$(1)\qquad \begin{cases} \dfrac{\partial v(t,x)}{\partial t} - D\dfrac{\partial^2 v(t,x)}{\partial x^2} + dv(t,x) = b(v(t-r,x)), & (t,x)\in\mathbb{R}_+\times\mathbb{R}, \\ v(s,x) = v_0(s,x), \quad s\in[-r,0],\ x\in\mathbb{R}, \end{cases}$$

which describes the population dynamics of a single species like the Australian blowflies [13, 14, 29, 30, 33, 43]. Here, $v(t,x)$ represents the mature population at time t and location x, $D>0$ the spatial diffusion rate of the mature species, $d>0$ the death rate, and $r>0$ the maturation delay. As described in [4, 25], $b:[0,\infty)\to(0,\infty)$ is the birth rate function, and is assumed to satisfy the following hypothesis:

- (H_1) Two constant equilibria $v_\pm$: $b(v_\pm)-dv_\pm=0$ for the homogeneous part of (1). We may take $v_-=0$ and thus $b(0)=0$. We further assume that v_- is unstable and v_+ is stable for the homogeneous part of (1). That is, $d-b'(0)<0$ and $d-b'(v_+)>0$.
- (H_2) The uni-modality condition: there is a $v_*\in(0,v_+)$ such that $b(\cdot)$ is increasing on $[0,v_*]$ and decreasing on $[v_*,+\infty)$. In particular, $b'(0)>0$ and $b'(v_+)<0$.
- (H_3) $b\in C^2[0,\infty)$ and $|b'(v)|\le b'(0)$ for $v\in[0,\infty)$.

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*Corresponding author.

Clearly, Hypothesis (H_1) implies that (1) is a mono-stable system, namely, one equilibrium of (1) is stable and the other one is unstable. The typical model for such mono-stable equations is the classic Fisher-KPP equation

$$v_t - v_{xx} = v(1-v).$$

Hypothesis (H_2) means that $b(v)$ is non-monotone for $v \in [0, v_+]$. As we shall see later, this leads to some oscillations for the solutions when the time-delay r is big.

There are also two featured examples for the equation (1) satisfying (H_1)-(H_3). One is the Nicholson's blowflies model by taking the birth rate function as

$$b(v) = pve^{-av}, \quad a > 0, \ p > 0, \tag{2}$$

where the constant equilibria are $v_- = 0$ and $v_+ = \frac{1}{a}\ln\frac{p}{d}$, and $b(v)$ is unimodal on $v \in [0, v_+]$ for $p/d > e$, and satisfies $|b'(v)| \le b'(0)$ for $v \in [0, \infty)$. This model was initially proposed by Gurney, Blythe, and Nisbet [11] based on the experiment data of blowflies by Nicholson [36, 37], see also the follow-up studies on wellposedness and asymptotic behavior of solutions in [13, 14, 19, 29, 30, 33, 43, 44].

The other is the Mackey-Glass model proposed in [27] (see also [12, 23, 29, 30] for further studies) by setting the birth rate function as

$$b(v) = \frac{pv}{1 + av^q}, \quad a > 0, \quad p > 0, \quad q > 1, \tag{3}$$

where $v_- = 0$ and $v_+ = \left(\frac{p-d}{da}\right)^{\frac{1}{q}}$. $b(v)$ is unimodal for $v \in [0, v_+]$ for $\frac{p}{d} > \frac{q}{q-1}$, satisfies $|b'(v)| \le b'(0)$ for $v \in [0, \infty)$.

Throughout this paper, naturally we assume that

$$\lim_{x \to \pm\infty} v_0(s, x) = v_\pm \quad \text{uniformly in } s \in [-r, 0]. \tag{4}$$

A traveling wave for (1) is a special solution to (1) of the form $\phi(x + ct) \ge 0$ with $\phi(\pm\infty) = v_\pm$:

$$\begin{cases} c\phi'(\xi) - D\phi''(\xi) + d\phi(\xi) = b(\phi(\xi - cr)), \\ \phi(\pm\infty) = v_\pm, \end{cases} \tag{5}$$

where $\xi = x + ct$, $' = \frac{d}{d\xi}$, and c is the wave speed. As summarized in [4, 25], there exists a number $c_* > 0$, called the minimum wave speed, which is uniquely determined by

$$c_*\lambda_* - D\lambda_*^2 + d = b'(0)e^{-\lambda_* c_* r} \quad \text{and} \quad c_* - 2D\lambda_* = -c_* r b'(0) e^{-\lambda_* c_* r}, \tag{6}$$

and when $c > c_*$, there exist two numbers $\lambda_2 > \lambda_1 > 0$ such that

$$c\lambda_i - D\lambda_i^2 + d = b'(0)e^{-\lambda_i cr}, \qquad \text{for } i = 1, 2, \tag{7}$$

and

$$c\lambda - D\lambda^2 + d > b'(0)e^{-\lambda cr}, \qquad \text{for } \lambda \in (\lambda_1, \lambda_2). \tag{8}$$

As showed in [6, 7, 12, 26, 47, 48], see also the summary in [4, 25], we have the following existence and uniqueness of the traveling waves as well as the property of oscillations:

• When $d \ge |b'(v_+)|$, the traveling wave $\phi(x + ct)$ exists uniquely (up to a shift) for every $c \ge c_* = c_*(r)$, where the time-delay r is allowed to be any number in $[0, \infty)$. If $0 \le r < \underline{r}$, where $\underline{r}$, given by

$$|b'(v_+)|\underline{r}e^{d\underline{r}+1} = 1, \tag{9}$$