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A DISCONTINUOUS RITZ METHOD FOR A CLASS OF CALCULUS OF VARIATIONS PROBLEMS

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Abstract. This paper develops an analogue (or counterpart) to discontinuous Galerkin (DG) methods for approximating a general class of calculus of variations problems. The proposed method, called the discontinuous Ritz (DR) method, constructs a numerical solution by minimizing a discrete energy over DG function spaces. The discrete energy includes standard penalization terms as well as the DG finite element (DG-FE) numerical derivatives developed recently by Feng, Lewis, and Neilan in [7]. It is proved that the proposed DR method converges and that the DG-FE numerical derivatives exhibit a compactness property which is desirable and crucial for applying the proposed DR method to problems with more complex energy functionals. Numerical tests are provided on the classical p-Laplace problem to gauge the performance of the proposed DR method.

Key words. Variational problems, minimizers, discontinuous Galerkin (DG) methods, DG finite element numerical calculus, compactness, convergence.

1. Introduction

In this paper we develop a numerical method using totally discontinuous piecewise polynomial functions for approximating solutions to the following problem from the calculus of variations: Find $u \in W_g^{1,p}(\Omega)$ such that

$$\mathcal{J}(u) \leq \mathcal{J}(v) \qquad \forall v \in W_g^{1,p}(\Omega), \tag{1}$$

where

$$\mathcal{J}(v) = \int_\Omega f(\nabla v, v, x)\,\mathrm{d}x \tag{2}$$

is the energy functional, $f : \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ is called the energy density, $\Omega \subset \mathbb{R}^d$ is an open bounded domain, and

$$W_g^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega) : u = g \text{ on } \partial\Omega\}.$$

If such a u exists, it is called a minimizer of $\mathcal{J}$ over $W_g^{1,p}(\Omega)$ and is written as

$$u \in \operatorname*{arg\,min}_{v \in W_g^{1,p}(\Omega)} \mathcal{J}(v). \tag{3}$$

Although the calculus of variations is an old field in mathematics, its growth and boundary have kept expanding because new applications arising from physics, differential geometry, image processing, materials science, and optimal control (just to name a few). Those problems are often formulated as calculus of variations problems, among them are the Brachistochrone problem [5], the minimal surface problem [6], and the Erickson energy for nematic liquid crystals [11].

Numerically solving those problems means to approximate the exact minimizer u of $\mathcal{J}$ over $W_g^{1,p}(\Omega)$ via a numerical approximation u_h. As expected, there are many methods for constructing an approximate solution u_h. The existing numerical methods can be divided into two categories: the indirect approach and the direct

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approach. The indirect approach is based on the fact that the minimizer u must satisfy, in some sense, the following Euler-Lagrange equation:

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(f_{\xi_i}(\nabla u, u, x) \right) = f_u(\nabla u, u, x) \qquad \forall x \in \Omega. \tag{4}$$

As equation (4) is a second order PDE in divergence (or conservative) form, it can be discretized using a variety of methods such as finite difference, finite element, discontinuous Galerkin and spectral method for constructing an approximate solution u_h. This indirect approach is often the preferred approach because of the wealthy amount of numerical methods available for discretizing PDEs. However, this approach does have two drawbacks. First, the Euler-Lagrange equation is only a necessary condition for a minimizer and it may not be a sufficient one. More information must be known about $\mathcal{J}$ in order to determine if the solution of the Euler-Lagrange equation indeed globally minimizes $\mathcal{J}$. Second, a discretization of the PDE may lose some important properties of the original energy functional, such as conservation or dissipation laws. On the other hand, the direct approach seeks an approximate solution u_h by first constructing a discrete energy functional $\mathcal{J}_h$ and then setting

$$u_h \in \arg\min_{v_h \in X_h} \mathcal{J}_h(v_h), \tag{5}$$

where X_h is a finite-dimensional space which approximates $W_g^{1,p}(\Omega)$. Since problem (5) is equivalent to a minimization problem in $\mathbb{R}^N$, a variety of algorithms (or solvers) can be employed to compute u_h. For example, we may minimize $\mathcal{J}_h$ by using a quasi-Newton algorithm or by first deriving the (discrete) Euler-Lagrange equation to $\mathcal{J}_h$ and then solving for u_h. The key issue of this approach is how to construct a "good" discrete energy functional $\mathcal{J}_h$ which can ensure the convergence of u_h to u. One important advantage of the direct approach is that a "good" discrete energy functional $\mathcal{J}_h$ will automatically preserve key properties of the original energy functional $\mathcal{J}$. For example, the discrete variational derivative method by Furihata and Matsuo for the KdV equation, nonlinear Schrödinger equations, and the Cahn-Hillard equation [9]; the Variational DGFEM method by Buffa and Ortner [2] for calculus of variations problems, and the finite element method by Nochetto *et al.* [11] for nematic liquid crystals all have such a trait.

Our goal in this paper is to develop a discontinuous Ritz (DR) framework for a class of variational problems described by (1). Our numerical method belongs to the direct approach and takes $X_h = V_h$ - the discontinuous Galerkin (DG) space consisting of totally discontinuous piecewise polynomial functions on a mesh $\mathcal{T}_h$ of Ω. We call our method a *discontinuous Ritz* method because it directly approximates problem (1). In the special case when

$$\mathcal{J}(v) = \frac{1}{2} a(v, v) - F(v),$$

and $a(\cdot, \cdot)$ is a symmetric and coercive bilinear form, problem (1) is known as the Ritz formulation of the following Galerkin (or weak) formulation: find $u \in V$ (which is assumed to be a Hilbert space) such that

$$a(u, v) = F(v) \qquad \forall v \in V.$$

As mentioned earlier, the key issue we face is to construct a "good" discrete energy functional $\mathcal{J}_h$. Since DG functions are discontinuous across element edges, two roadblocks arise when creating a discrete energy functional $\mathcal{J}_h$ that makes sense