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AN ITERATIVE APPROACH FOR CONSTRUCTING IMMERSED FINITE ELEMENT SPACES AND APPLICATIONS TO INTERFACE PROBLEMS

CHENG WANG, PENGTAO SUN, AND ZHILIN LI

Abstract. In this paper, an iterative approach for constructing immersed finite element spaces is developed for various interface conditions of interface problems involving multiple primary variables. Combining such iteratively constructed immersed finite element spaces with the distributed Lagrange multiplier/fictitious domain (DLM/FD) method, we further present a new discretization method that can uniformly solve general interface problems with multiple primary variables and/or with different governing equations on either side of the interface, including fluid-structure interaction problems. The strengths of the proposed method are shown in the numerical experiments for Stokes- and Stokes/elliptic interface problems with different types of interface conditions, where, the optimal or nearly optimal convergence rates are obtained for the velocity variable in H^1, L^2 and L^∞ norms, and at least 1.5-th order convergence for the pressure variable in L^2 norm within few number of iterations. In addition, numerical experiments show that such iterative process uniformly converges and the number of iteration is independent of mesh ratios and jump ratios.

Key words. Immersed finite element (IFE) method, fictitious domain method, Lagrange multiplier, iterative process, interface problems, fluid-structure interactions (FSI).

1. Introduction

Physical phenomena in a domain consisting of multiple materials or fluids with an interface are often modeled by differential equations with discontinuous coefficients which are often called interface problems. Solutions to interface problems are often required to satisfy jump conditions across the material interfaces in addition to the pertinent differential equations and the related boundary conditions. In general, interface problems require the governing differential equations at the common interface to share not only the common value of primary variable (Dirichlet-type interface condition) but also the common flux (Neumann-type interface condition).

Due to its simplicity in mesh generation (one single uniform mesh usually works well), the body-unfitted mesh method becomes more promising in solving interface problems with moving interfaces possessing sophisticated and irregular shapes. Among the existing body-unfitted mesh methods, for example, the extended finite element method (XFEM), also known as a generalized finite element method (GFEM) in which enrichment functions are added near the interface [32]; unfitted discontinuous Galerkin methods with penalties [27]; unfitted finite element method based on the Nitsche's method in [15] and etc., the immersed finite element (IFE) method, which was originally proposed in [20, 21] for solving elliptic interface problems, turns out to be the most accurate and efficient because it can avoid the smearing of the sharp interface without introducing any local mesh enrichment, and maintains second-order accuracy by incorporating the known jump conditions at the interface into the finite element space.

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Since its beginning, the IFE method has been mostly applied to the elliptic interface problem [21, 23, 16, 17, 1, 18, 19] due to the simplicities of both governing equation and interface conditions. However, the IFE method still suffers from applying to the Stokes interface problem because of its sophisticated governing equations (an intrinsic saddle-point structure) and complicated interface conditions in contrast with the elliptic interface problem, making the stable mixed finite element (Stokes-pair) very difficult to be defined and analyzed in the immersed finite element space. So far, only a mixed IFE Q_1/Q_0 is designed for the following Stokes interface problem (1)-(8) [2], however, the discontinuous Galerkin method has to be relied on in order to stabilize the mixed IFE Q_1/Q_0 since Q_1/Q_0 is not a stable Stokes-pair.

$$-\nabla\cdot(\beta_1\nabla \boldsymbol{u}_1)+\nabla p_1=\boldsymbol{f}_1, \qquad \text{in } \Omega_1, \tag{1}$$
$$\nabla\cdot\boldsymbol{u}_1=0, \qquad \text{in } \Omega_1, \tag{2}$$
$$-\nabla\cdot(\beta_2\nabla \boldsymbol{u}_2)+\nabla p_2=\boldsymbol{f}_2, \qquad \text{in } \Omega_2, \tag{3}$$
$$\nabla\cdot\boldsymbol{u}_2=0, \qquad \text{in } \Omega_2, \tag{4}$$
$$\boldsymbol{u}_1=\boldsymbol{u}_2, \qquad \text{on } \Gamma, \tag{5}$$
$$(\beta_1\nabla\boldsymbol{u}_1-p_1\boldsymbol{I})\boldsymbol{n}_1+(\beta_2\nabla\boldsymbol{u}_2-p_2\boldsymbol{I})\boldsymbol{n}_2=\boldsymbol{w}, \qquad \text{on } \Gamma, \tag{6}$$
$$\boldsymbol{u}_1=0, \qquad \text{on } \partial\Omega_1\backslash\Gamma, \tag{7}$$
$$\boldsymbol{u}_2=0, \qquad \text{on } \partial\Omega_2\backslash\Gamma, \tag{8}$$

where, $\Omega=\Omega_1\cup\Omega_2\subset\mathcal{R}^d$ as shown in Figure 1, and the immersed interface $\Gamma=\partial\Omega_2$ is generally a closed curve that divides the domain Ω into an interior region Ω_2 and an exterior region Ω_1, and splits an arbitrary function $\boldsymbol{\phi}\in L^2(\Omega)$ to be $\boldsymbol{\phi}|_{\Omega_i}=\boldsymbol{\phi}_i\ (i=1,2)$, where the subscripts 1 and 2 indicate the restrictions to the corresponding subdomain, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ stand for the unit outward normal vectors on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. We assume $\boldsymbol{f}_i\in\left(L^2(\Omega_i)\right)^d$, $\boldsymbol{w}\in\left(H^{1/2}(\Gamma)\right)^d$. The coefficient $\beta(\boldsymbol{x})$ and the source term $\boldsymbol{f}(\boldsymbol{x})$ may exhibit discontinuities across Γ, but have smooth restrictions $\beta_1(\boldsymbol{x})$, $\boldsymbol{f}_1(\boldsymbol{x})$ in Ω_1 and $\beta_2(\boldsymbol{x})$, $\boldsymbol{f}_2(\boldsymbol{x})$ in Ω_2. In addition, the following regularity properties are held for the Stokes interface problem (1)-(8) if the interface Γ is of class C^2 [31]

$$\boldsymbol{u}\in\boldsymbol{X}:=(H^s(\Omega))^d\cap(H^2(\Omega_1\cup\Omega_2))^d, \quad p\in\boldsymbol{Y}:=L^2(\Omega)\cap H^1(\Omega_1\cup\Omega_2), \tag{9}$$

where, $1<s<1.5$.

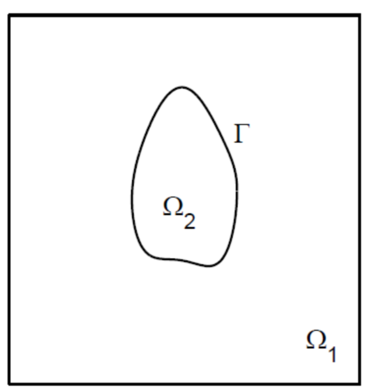

FIGURE 1. Graphical depiction of the domain with an immersed interface.