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RELIABLE AND EFFICIENT A POSTERIORI ERROR ESTIMATES OF DG METHODS FOR A SIMPLIFIED FRICTIONAL CONTACT PROBLEM

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Abstract. A posteriori error estimators are studied for discontinuous Galerkin methods for solving a representative elliptic variational inequality of the second kind, known as a simplified frictional contact problem. The estimators are derived by relating the error of the variational inequality to that of a linear problem. Reliability and efficiency of the estimators are theoretically proved.

Key words. Elliptic variational inequality, discontinuous Galerkin method, a posteriori error estimators, reliability, efficiency.

1. Introduction

For more than three decades, adaptive finite element method (AFEM) has been an active research field in scientific computing. As an efficient numerical approach, it has been widely used for solving a variety of differential equations. Each loop of AFEM consists of four steps:

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}.$$

That is, in each loop, we first solve the problem on a mesh, then use a posteriori error estimators to mark those elements to be refined, and finally, refine the marked elements and get a new mesh. We can continue this process until the estimated error satisfies certain smallness criterion. The adaptive finite element method can achieve required accuracy with lower memory usage and less computation time.

A posteriori error estimators are computable quantities that provide the contribution of error on each element to the global error. They are used in adaptive algorithms to indicate which elements need to be refined or coarsened. To capture the true error as precisely as possible, they should have two properties: reliability and efficiency ([1, 4]). Hence, obtaining reliable and efficient error estimators is the key for successful adaptive algorithms. A variety of a posteriori error estimators have been proposed and analyzed in the literature. Many error estimators can be classified as residual type or recovery type ([1, 4]). Various residual quantities are used to capture lost information going from u to u_h, such as residual of the equation, residual from derivative discontinuity and so on. Another type of error estimators is gradient recovery, i.e., $||G(\nabla u_h) - \nabla u_h||$ is used to approximate $||\nabla u - \nabla u_h||$, where a recovery operator G is applied to the numerical solution u_h to rebuild the gradient of the true solution u. A posteriori error analysis has been well established for standard finite element methods for solving linear partial differential equations, and we refer the reader to [1, 4, 30].

Due to the inequality feature, it is more difficult to develop a posteriori error estimators for variational inequalities (VIs). However, numerous articles can be found on a posteriori error analysis of finite element methods for the obstacle

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problem, which is an elliptic variational inequality (EVI) of the first kind, e.g., [5, 15, 24, 26, 29, 34]. In [11], Braess demonstrated that a posteriori error estimators for finite element solutions of the obstacle problem can be derived by applying a posteriori error estimates for an associated linear elliptic problem. For VIs of the second kind, in [7, 8, 9, 10], the authors studied a posteriori error estimation and established a framework through the duality theory, but the efficiency was not completely proved. In [31], the ideas in [11] were extended to give a posteriori error analysis for VIs of the second kind. Moreover, a proof was provided for the efficiency of the error estimators.

In recent years, thanks to the flexibility in constructing feasible local shape function spaces and the advantage to capture non-smooth or oscillatory solutions effectively, discontinuous Galerkin (DG) methods have been widely used for solving various types of partial differential equations. When applying h-adaptive algorithm with standard finite element methods, one needs to choose the mesh refinement rule carefully to maintain mesh conformity and shape regularity. In particular, hanging nodes are not allowed without special treatment. For discontinuous Galerkin methods, the approximate functions are allowed to be discontinuous across the element boundaries, so general meshes with hanging nodes and elements of different shapes are acceptable. Advantages of DG methods include the flexibility of mesh-refinements and construction of local shape function spaces (hp-adaptivity), and the increase of locality in discretization, which is of particular interest for parallel computing. A historical account on the development of DG methods can be found in [16]. In [2, 3], Arnold et al. established a unified error analysis of nine DG methods for elliptic problems and several articles provided a posteriori error analysis of DG methods for elliptic problems (e.g. [6, 12, 14, 22, 25, 27]). Carstensen et al. presented a unified theory for a posteriori error analysis of DG methods in [13]. In [32], the authors extended ideas of the unified framework about DG methods for elliptic problems presented in [3] to solve the obstacle problem and a simplified frictional contact problem, and obtained a priori error estimates, which reach optimal order for linear elements. In [33], reliable a posteriori error estimators of the residual type were derived for DG methods for solving the obstacle problem, and efficiency of the estimators is theoretically explored and numerically confirmed. A posteriori error analysis of DG methods for the obstacle problem was also studied in [20].

A posteriori error analysis of DG methods has not been well-studied for variational inequalities of the second kind. In this paper, we explore this topic and study a posteriori error estimates of DG methods for solving a representative elliptic variational inequality of the second kind, namely, the simplified frictional contact problem. The estimators are derived by relating the error of the variational inequality to that of a corresponding linear problem. Furthermore, the reliability and efficiency of the estimators are theoretically proved. Even though we only consider the residual type error estimators in this paper, an analysis for gradient recovery type error estimation can be obtained by the techniques used in this paper and the standard argument of gradient recovery type error analysis for the second order elliptic equations.

The paper is organized as follows. In Section 2, we introduce the variational inequality problem and the DG schemes for solving it. Then we derive reliable a posteriori error estimators of residual type for the DG methods of the simplified frictional contact problem in Section 3. Finally, we prove efficiency of the proposed error estimators in Section 4.