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MODULAR NONLINEAR FILTER BASED TIME RELAXATION SCHEME FOR HIGH REYNOLDS NUMBER FLOWS

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This paper is dedicated to Professor William Layton's 60th birthday.

Abstract. In this article, we propose and develop a time relaxation implementation of the modular nonlinear filter model of [21]. A complete numerical analysis of the scheme, that includes the computability of its numerical solutions, its stability, and velocity error estimates, is given. This is followed by 2D and 3D numerical experiments that show the advantage of the proposed scheme.

Key words. Time relaxation, finite element, nonlinear filtering, deconvolution, Navier-Stokes equations

1. Introduction

The range of size of the velocity eddies is very wide, especially in simulation with higher Reynolds number. Based on the Kolmogorov theory [16], the computations have to be done on a very fine mesh to be able to capture all the persistent eddies and these proper numerical computations are not feasible with the current computer power. For this reason, numerical regularization and computational stabilizations have been explored in computational fluid dynamics, [4, 26, 15]. Herein, we study a regularization that has been proposed by Adams, Stoltz and Kleiser [1, 2, 29, 30]. Let $\mathbf{u}$ represent the fluid velocity, h the characteristic mesh width, and $\delta = O(h)$ a chosen length scale, and $\mathbf{u}^{'}$ denote some representation of the part of $\mathbf{u}$ varying over length scales $< O(\delta)$, i.e., the fluctuating part of $\mathbf{u}$. This will be made specific in Section 2 and 3. The fluid regularization model that we consider, was obtained by adding a time regularization term, $\chi\mathbf{u}^{'}$ to the Navier-Stokes equations (NSE)

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} + \chi\mathbf{u}^{'} = \mathbf{f},\ x \in \Omega\,, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0,\ x \in \Omega\,. \tag{2}$$

The term $\chi\mathbf{u}^{'}$ is a linear, lower order term and it is intended to drive unresolved velocity scales to zero exponentially fast. With that aim, $\chi > 0$, and has units of $1/time$. The regularizations of this type have been extensively studied in the literatures. Adams, Kleiser and Stoltz have performed numerical tests of this time relaxation model on compressible flows with shocks and on turbulent flows, [28, 29, 30]. Guenanff [11] performed studies on aerodynamic noise. Rosenau [25], Schochet and Tadmor [27] did studies of (1)-(2) in which the time relaxation model was developed from a regularized Chapman-Enskog expansion of conservation laws. In [22], it was shown that at high Reynolds number, solutions to (1)-(2), possess an energy cascade which terminates at the mesh scale δ with the proper choice of relaxation coefficient χ. Also, the joint helicity/energy cascade was investigated in [20]. A standard continuous finite element analysis of the model (1)-(2) was performed in [10].

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In [24], following the work from [6], it was also studied a continuous finite element discretization of (1)-(2) that incorporated three ideas. First idea was to use incompressible filter (i.e. a Stokes type of filter problem) for better consistency outside of the periodic domains. Second idea was the efficient implementation of linearization of Baker [3], that allows to solve for only one linear system per time step with second order of accuracy. The third idea was the stabilization in time that is natural for this linearization and which was first introduced in [17].

The goal of this paper is to present the implementation of the time-relaxation regularization through the nonlinear filter stabilization method of [21]. The attractive feature of the modular adaptive nonlinear filter model [21] is that it allows one to incorporate a desired eddy-viscosity model into the legacy codes by solving an additional Stokes-Darcy type system, as mentioned in [5]. The idea has been further extended to improve the Leray-α model in [7], and a first order, computationally efficient implementation has been recently reported in [9]. The proposed scheme is also easy to incorporate into the existing codes. It requires solution of the Stokes like system (or just an elliptic problem, since Laplace type of filtering showed comparable results to Stokes for the few performed numerical experiments), and changing the coefficient in the mass matrix.

This article is organized the following way. In Section 2 we give a precise definition of the discrete nonlinear filtering operator and of the generalized fluctuation $\mathbf{u}^{'}$. We also give preliminaries about the finite element framework. Section 3 gives the scheme and its unconditionally stability. In Section 4 the finite element convergence error analysis is presented. In Section 5, we present 2D and 3D numerical tests that show the effectiveness of the nonlinear filters for the time relaxation model.

2. Notation and Preliminaries

In order to discuss the effects of the regularization we introduce the following notation. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ denotes the norm in H^k. For functions $\mathbf{v}(\mathbf{x},t)$ defined on the entire time interval $(0,T)$, we define

$$\|\mathbf{v}\|_{\infty,k} := \sup_{0<t<T} \|\mathbf{v}(\cdot,t)\|_k\,, \qquad \text{and} \qquad \|\mathbf{v}\|_{m,k} := \left(\int_0^T \|\mathbf{v}(\cdot,t)\|_k^m\,dt\right)^{1/m}.$$

The following function spaces are used in the analysis:

$$\begin{aligned}
\text{Velocity Space} \quad &- \quad X := H_0^1(\Omega)\,,\\
\text{Pressure Space} \quad &- \quad P := L_0^2(\Omega) = \left\{ q \in L^2(\Omega)\,:\, \int_\Omega q\,d\Omega = 0 \right\},\\
\text{Divergence} - \text{free Space} \quad &- \quad Z := \left\{ \mathbf{v} \in X : \int_\Omega q\,\nabla\cdot\mathbf{v}\,d\Omega \,=\, 0,\ \forall\, q \in P \right\}.
\end{aligned}$$

We denote the dual space of X as X', with norm $\|\cdot\|_{-1}$.
Let $\Omega \subset \mathbb{R}^{\acute{d}}$ $(\acute{d} = 2,3)$ be a polygonal domain and let T_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrons (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\Omega = \cup K;\ \ K \in T_h.$$