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HYPERBOLIC PDES ACROSS GASDYNAMICS, PLASTICITY, AND GRANULAR PLASTICITY

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Dedicated to Professor William J. Layton on the occasion of his 60^{th} *birthday*

Abstract. This expository paper is presented in celebration of Professor Layton's 60th birthday. In mathematics genealogy, he is the gg-grandson of Göttingen's Ludwig Prandtl (1875-1953), who contributed enormously to both fluid and solid mechanics. The Prandtl-Meyer fan is a simple tool to visualize solutions of hyperbolic PDEs in the context of gasdynamics. In later years, Prandtl adapted the fan to visualize solutions of hyperbolic PDEs in the context of plasticity in metals. By 1960, the fan had been used by Sokolovskii in the context of granular plasticity. Because the last introduces logarithms into plasticity's equivalent of the Riemann invariant, granular materials exhibit unexpected behaviors that are being ignored by the construction community with costly consequences. Interestingly, one of the consequences for soil is analogous to the Rankine-Hugoniot relation in gasdynamics.

Key words. Prandtl fan, gasdynamics, plasticity, granular plasticity, Euclidean

1. Introduction

Hyperbolic partial differential equations (PDEs) provide the common framework that facilitated work by Prandtl, his students, and his colleagues as they moved among the disciplines of gasdynamics, plasticity, and granular plasticity:

- Gasdynamics involves compressibility, and it usually involves high speed flow where shocks and other compressibility effects are evident.
- Plasticity is contrasted with elasticity; that is, an elastic object regains its shape after a load is removed, but in plasticity, it does not. Plasticity applies to metal forming.
- Granular plasticity describes gritty materials that exhibit internal friction, such as salt, soil, concrete, and exotic metals[5]. Examples of its application are provided later.

This paper explores analogies among these disciplines and their PDEs. Gasdynamicists understand the difference between the Euler equation and the Navier-Stokes equation. The former is inviscid, and the latter includes viscous effects. Similarly, plasticity is inviscid, and granular plasticity includes viscous effects; that is,

Euler Eq : Navier-Stokes Eq :: plasticity : granular plasticity

This analogy ignores boundary friction in plasticity, which must be countered by lubricants during metal forming. In contrast, the friction of granular plasticity is internal friction everywhere within the material.

The Euler and Navier-Stokes equations are evolution equations, and they are solved by marching in time. Plasticity uses strain rate, not time, but strain became a point of dispute between plasticity and granular plasticity. This dispute, which

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is associated with plasticity's Normality Rule, will be discussed. Therefore, time is avoided, and steady-state equations are used here. For steady-state inviscid PDEs, consider this analogy,

elliptic : hyperbolic :: subsonic : supersonic :: elastic : plastic

Graphically, these analogies are emphasized by the Prandtl-Meyer fan and its adaptation across the disciplines.

2. Gasdynamics

Before the supercomputer, numerical solution of conservation laws and evolution equations was usually impractical.

2.1. Since 1977: Evolution Equations. Computing caused a revolution in mathematics, but Seymour Cray, the person at the center of the revolution, knew nothing about the math of gasdynamics other than the importance of linear algebra... fast linear algebra.

Before building his Cray 1 in 1977, he had humiliated IBM by breaking the MFLOP barrier with the CDC 6600. This prompted IBM CEO Thomas J. Watson to write a famous, and angry, memo to his employees [9]:

Last week, Control Data... announced the 6600 system. I understand that in the laboratory developing the system there are only 34 people including the janitor... Contrasting this modest effort with our vast development activities, I fail to understand why we have lost our industry leadership position by letting someone else offer the world's most powerful computer.

With the advent of high-performance computing in 1977, long after World War II, time-dependent conservation laws became the focus. In gasdynamics, these conservation laws are called the Euler equations, and they express conservation of mass, momentum, and energy. As computer power further increased, researchers added chemistry or tackled the Navier-Stokes equations. The latter disturb conservation with viscosity, which gives the dissipative, but interesting, effects of boundary layers and turbulence.

2.1.1. Example: Space Shuttle Orbiter. The first orbital flight of the Space Shuttle occurred in 1981. The orbiter's behavior differed greatly from wind tunnel predictions [1]:

Flight experience with the shuttle has indicated a much higher pitching moment at hypersonic speeds than predicted; this has required the body flap deflection for trim to be more than twice that predicted.

Adding drama, Bertin and Cummings [3] emphasized that the required flap angle was...

...close to the limit of possible deflection.

The difference was explained because an early Cray computer, shared across America by phone lines, enabled the inclusion of high temperature air chemistry within a flow calculation by Maus and others [15]. This episode became a famous textbook example decisively showing that computations could provide an alternative to wind tunnels [1].

At the time, each and every bit of memory required a worker to thread copper wires through a tiny iron donut. With memory and speed limitations, early computations made compromises with (1) complex physics, (2) complex geometry, and (3) time evolution. Computers of the early 1980's required users to choose one from these three desired capabilities, but a focused industry made rapid progress over the next two decades.