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AN APPROXIMATE ALGORITHM TO SOLVE LINEAR SYSTEMS BY MATRIX WITH OFF-DIAGONAL EXPONENTIAL DECAY ENTRIES

QIANGSHUN CHANG, YANPING LIN, AND SHUZHAN XU

Abstract. We present an approximate algorithm to solve only one variable out of a linear system defined by a matrix with off-diagonal exponential decay entries (including the practically most important class of band limited matrices) via a sub-linear system. This approach thus enables us to solve any subset of solution variables. Parallel implementation of such approximate schemes for every variable enables us to solve the linear system with computational time independent of the matrix size.

Key words. Linear equation, numerical solution, sub-linear system, decomposition.

1. Introduction and motivations

How to solve a large linear system $Ax = f$ is a key topic of practical importance. Direct approaches (like Gauss elimination and various decomposition schemes, such as LU and QR, are used mainly for small matrices) are theoretically precise yet practically forbidden for large linear systems in general due to computational cost. Iterative approaches (such as Jacobi, Gausss-Seidel, SOR and CG iteration, [8, 19]) and multigrid methods [17] are approximate methods and basically "the practical schemes". For iterative methods, the matrix condition number (defined as $cond(A) = \max_{||x||_2=1} \frac{||Ax||_2}{\|x\|_2} = \frac{\lambda_{max}}{\lambda_{min}}$, where λ_{max} and λ_{min} are the maximum and the minimum eigen-values of A respectively) usually decides their convergence rates and the matrix "sparsity" (i.e. number of non-zero entries in A) decides their computation costs. Iteration by nature, the convergence speed of multigrid methods is independent of the matrix condition numbers though [12, 17]. For linear systems derived from numerical differential equations via finite difference [16] or finite element [14], the matrix sizes are generally decided by the size of domain and the approximation accuracies required. Domain decomposition ([12, 13], to split the original problem into problems with smaller domains) and preconditioning ([11], to transform the matrix for better condition number) studies are trying to deal with large matrices and poor matrix condition numbers. They are usually "geometrical" methods linked to the original problem. Algebraic multigrid, a special iterative method [17], is based on the algebraic properties of the final linear systems mainly.

Heading in a different direction, there are also lot of recent progress trying to reduce the number of computations and to split the matrices (i.e. different decomposition schemes). Almost purely algebra in nature, these works try to solve the linear system efficiently by looking at the matrix structure directly. The first approach is the work on semiseparable matrices (also been used for symmetric eigen-value problems) [18]. A symmetric matrix can be transformed into tridiagonal, semiseparable or with diagonal plus semiseparable form (free diagonal choice) via orthogonal similarity or Lanczos-like reduction. Efficient algorithms can then be

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devised accordingly [18]. For example, Crout algorithm can be applied to solve the tridiagonal form and efficient QR-factorization approach can be used to solve the semiseparable form. Another approach is H-matrices [2] with the aim of enabling matrix operations in almost linear complexity. The key technique is to applying local matrix approximation via matrices that is product of two vectors. Based on Taylor series analysis of kernel $\log|x-y|$, local matrix approximation can be applied with low rank approximation of matrix blocks. Almost linear complexity algorithm can then be devised via cluster tree partition technically. We emphasize that these techniques, easier to apply with efficiency for small matrices, can be applied on top of our scheme since our approach is to decompose the system into smaller systems first. Future work combining these ideas with decomposition most likely will further refine our algorithm.

Last but not least, we mention algebraic Schwarz or algebraic domain decomposition methods which are mostly related to our work [1, 15, 20]. They are iterative approaches via Schwarz alternating in algebraic form and can actually be viewed more clearly in its elliptic problem theoretical background. H. A. Schwarz's study of Dirichlet problem on overlapping regions provided the fundamental alternating solution approach (numerically a different way of iteration). The elegant and insightful analysis of P. L. Lions and O. B. Widlund [5, 9, 10] are recent reinterpretations and further developments (for example parallel algorithms) of this classical direction. Amazingly enough, our very first feeling is that these projection analysis techniques might be borrowed and modified for the iterative turbo decoding analysis. It is also interesting to recall that turbo codes was invented by C. Berrou, A. Glavieux and P. Thitimajshima in 1993 from France (see references in [7]). Secondly and most importantly, we feel that the connection between iterative algebraic domain decomposition and our direct approach to be presented deserves serious further investigation (in particular the Dirichlet problem counter part analysis).

In our effort starting with algebraic multigrid looking for schemes to make the matrix to have better condition number and to decompose large linear systems, we found a fast approximate algorithm capable of breaking the matrix size (for special classes of matrices of course) to be presented here. This algorithm seems can be used in many areas beyond numerical partial differential equations. It thus justifies an independent paper. Our main contribution is the algorithm capable of solving a single variable by solving a smaller linear system in some special large linear systems with controllable error. Even can be further elaborated and extended, we mainly study matrices with off-diagonal exponential decay entries for simplicity and practical efficiency. Counter examples show easily that our algorithm is not valid for all matrices. For practical implementation concerns, we also present exact conditions for a matrix to be with off-diagonal exponentially decay entries.

Let us look at some simple symmetric positive definite matrix examples with exponential decay entries to build up our intuitions for further analysis. For $A = \left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right)$, where $a > 0$, $d > 0$, $a >> c$, and $d >> c$. Linear equation $Ax = b_1$ has solutions $x = (db_1 - cb_2)/(ad - c^2)$ and $x_2 = (ab_2 - cb_1)/(ad - c^2)$. We can see that $\lim_{c\to 0} x_1 = \dfrac{b_1}{a}$ and $\lim_{c\to 0} x_2 = \dfrac{b_2}{d}$. That is solutions of $\left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$ are close to solutions of $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$, which is a compressed form with c set to zero. Let's ponder on this observation and extend our analysis to a larger matrix. Suppose $A = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right)$ is symmetric positive definite and with off-diagonal exponential