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APPROXIMATION OF THE LONG-TERM DYNAMICS OF THE DYNAMICAL SYSTEM GENERATED BY A 3D NS-α SYSTEM WITH PHASE TRANSITION

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Abstract. In this article we study an approximate model for a binary fluid flow in a three-dimensional bounded domain. The governing equations consist of the Allen–Cahn equation for the order (phase) parameter ϕ coupled with the Navier–Stokes-α (NS-α) system for the velocity u. We discretize these equations in time using the implicit Euler scheme and we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

Key words. Navier–Stokes-α, phase transition, attractors, implicit Euler scheme, Gronwall Lemma.

1. Introduction

It is well known that the incompressible Navier–Stokes equations govern the motion of single-phase fluids, such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is, fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [23]. For instance, this approach is used in [3] to describe cavitation phenomena in a flowing liquid. The model consists of the Navier–Stokes equations coupled with the phase-field system, [4, 23, 22, 24]. In the isothermal compressible case, the existence of a global weak solution is proved in [18]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity u and the order parameter ϕ. This system can be written as a Navier–Stokes equation coupled with a convective Allen–Cahn equation, [23]. The associated initial and boundary value problem was studied in [23], in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor $\mathcal{A}$. They also established the existence of an exponential attractor $\mathcal{E}$. This entails that $\mathcal{A}$ has a finite fractal dimension, which is estimated in [23] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated, although some important issues remain unresolved, [40]. In the case of binary fluids, the analysis is even more complicated and the mathematical study is still at its infancy, as noted in [23].

In this article we study an approximate model for a binary fluid flow in a three-dimensional bounded domain. The model is derived from the 3D coupled Allen–Cahn–Navier–Stokes system by substituting the 3D Navier–Stokes system with the 3D NS-α equations. This model can be considered as a regularized approximation of the 3D coupled Allen–Cahn–Navier–Stokes system, depending on a small positive parameter $\alpha > 0$, where in some terms, the unknown velocity function v is replaced

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by a smoother function u, solution of the elliptic system $v = u - \alpha^2 \Delta u$. For $\alpha = 0$, the model reduces to the exact 3D coupled Allen–Cahn–Navier–Stokes system.

Since the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D coupled Allen–Cahn–Navier–Stokes system is not proved yet, the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D coupled Allen–Cahn–Navier-Stokes system. This situation is the same for the 3D Navier–Stokes systems. Using regular approximation equations to study the classical 3D Navier–Stokes systems has become an effective tool both from the numerical and the theoretical point of views. It is well-known that direct numerical simulation of the 3D NSE for many physical applications with high Reynolds number flows is intractable even using state-of-the-art numerical methods on the most advanced supercomputers available nowadays. Recently, many applied mathematicians have developed regularized turbulence models for the 3D NSE as an attempt to overcome this simulation barrier. Their aim is to capture the large, energetic eddies without having to compute the smallest dynamically relevant eddies, by instead modelling the effects of small eddies in terms of the large scales in the 3D NSE. Since 1998, many such regularized models have been proposed, tested and investigated from both the numerical and the mathematical point of views. Among these models, one can find the globally well-posed 3D Navier–Stokes-α (NS-α) equations (also known as the viscous Camassa–Holm equations and Lagrangian averaged Navier–Stokes-α model), the 3D Leray-α models, the modified 3D Leray-α models, the simplified 3D Bardina models, the 3D Navier–Stokes–Voight (NSV) equations, and their inviscid counterparts. As noted in [14], it was demonstrated analytically and numerically that the NS-α model gives a good approximation in the study of many problems related to turbulent flows. In particular, it was found that the explicit steady analytical solution of the NS-α model compares successfully with empirical and numerical experiment data for a wide range of Reynolds numbers in turbulent channel and pipe flows, [14]. Let us recall that the inviscid 3D NS-α equations were first proposed in [19]. As described in [19, 33], the 3D NS-α equations are a systems of partial differential equations for the mean velocity in which a nonlinear dispertive mechanism filters the small scales. As such, the 3D NS-α equations serve as an appropriate model for turbulent flows and a suitable approximation of the 3D Navier–Stokes, as documented in [9, 11, 10, 12, 31, 30, 33, 25, 26, 27, 13, 29]. A successful comparison with data for time-averaged quantities for a wide range of Reynolds numbers in turbulent channel and pipe flows was done in [9, 11]. Further studies of the 3D NS-α models in the context of turbulence modeling appear in [15, 32, 36]. Analytical studies of the global existence, uniqueness and regularity of solutions to the 3D NS-α system are performed in [19] in the case of periodic boundary conditions. Some existence and uniqueness results are also established in [33, 7, 8, 6]. In [33], the authors prove the global well-posedness and regularity of the 3D NS-α equations in a bounded domain with a non-slip boundary condition. A non-autonomous NS-α model is considered in [7], where the authors study the asymptotic behavior of the solutions of a 3D NS-α with delay forces. They prove the existence of a pullback and forward attractors for the model. The stochastic version is also studied in [6].

Motivated by the above works and the fact that a full mathematical theory of the 3D coupled Allen–Cahn–Navier–Stokes system is still lacking, the author in [34] studied an approximate model for a binary fluid flow in a three dimensional bounded domain. The governing equations consist of the Allen–Cahn equations