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FULLY COMPUTABLE ERROR BOUNDS FOR EIGENVALUE PROBLEM

QICHEN HONG, HEHU XIE, MEILING YUE AND NING ZHANG

This paper is dedicated to Prof. Benyu Guo

Abstract. This paper is concerned with the computable error estimates for the eigenvalue problem which is solved by the general conforming finite element methods on the general meshes. Based on the computable error estimate, we can give an asymptotically lower bound of the general eigenvalues. Furthermore, we also give a guaranteed upper bound of the error estimates for the first eigenfunction approximation and a guaranteed lower bound of the first eigenvalue based on computable error estimator. Some numerical examples are presented to validate the theoretical results deduced in this paper.

Key words. Eigenvalue problem, computable error estimate, guaranteed upper bound, guaranteed lower bound, complementary method.

1. Introduction

This paper is concerned with the computable error estimates for the eigenvalue problem by the finite element method. As we know, the priori error estimates can only give the asymptotic convergence order. The a posteriori error estimates are very important for the mesh adaption process. Interested readers can refer to [2, 6, 7, 8, 27, 28, 34] and the references cited therein for more information about the a posteriori error estimate for the partial differential equations by the finite element method.

This paper is to give computable error estimates for the eigenpair approximations. We produce a guaranteed upper-bound error estimate for the first eigenfunction approximation and then a guaranteed lower bound of the first eigenvalue. The approach is based on complementary energy method from [15, 27, 28, 31, 32] coupled with the upper and lower bounds of the eigenvalues by the conforming and nonconforming finite element methods. The first eigenvalue is the key information in many practical applications such as Friedrichs, Poincaré, trace and similar inequalities (cf. [29]). Thus the two-sided bounds of the first eigenvalue of the partial differential operators are very important. Furthermore, the proposed computable error estimates are asymptotically exact for the general eigenpair approximations which are obtained by the conforming finite element method. Based on this property, we can provide asymptotically lower bounds for general eigenvalues by the finite element method. The most important feature and contribution of this paper are that the method can also provide the reasonable accuracy even on the general regular meshes which is different from the existed methods (cf. [3, 10, 11, 16, 18, 19, 22, 23, 24, 37, 38]).

It is well known that the numerical approximations by the conforming finite element methods are upper bounds of the exact eigenvalues. Recently, how to obtain the lower bounds of the desired eigenvalues is a hot topic since it has many applications in some classical problems [3, 10, 11, 16, 18, 19, 22, 23, 24, 29, 30, 33, 37,

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38]. Up to now, there have developed the nonconforming finite element methods [3, 16, 18, 19, 22, 37, 38], interpolation constant based methods [10, 11, 23, 24] and computational error estimate methods [29, 30, 33]. The nonconforming finite element methods can only obtain the asymptotically lower bounds with the lowest order accuracy. The interpolation constant method can only obtain the efficient guaranteed lower bounds only on the quasi-uniform meshes since the accuracy is determined by the global mesh size. The complementary method in [31, 32] for computing the a posteriori error estimate of the finite element method gives a clue to this paper.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the eigenvalue problem and the corresponding basic error estimates. The computable error estimates for the eigenfunction approximations and the corresponding upper-bound properties are given in Section 3. In Section 4, lower bounds of eigenvalues are obtained based on the results in Section 3. Some numerical examples are presented to validate our theoretical analysis in Section 5. Some concluding remarks are given in the last section.

2. Finite element method for eigenvalue problem

This section is devoted to introducing some notation and the finite element method for eigenvalue problem. In this paper, the standard notation for Sobolev spaces $H^s(\Omega)$ and $H(\mathrm{div};\Omega)$ and their associated norms and semi-norms [1] will be used. We denote $H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace. The letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences in the paper.

For simplicity, this paper is concerned with the following model problem: Find (λ, u) such that

$$\left\{\begin{array}{rcll} -\Delta u + u &=& \lambda u, & \text{in } \Omega, \\ u &=& 0, & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset \mathcal{R}^d$ $(d = 2, 3)$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and Δ denotes the Laplacian operator. We will find that the method in this paper can easily be extended to more general eigenvalue problems.

In order to use the finite element method to solve the eigenvalue problem (1), we need to define the corresponding variational form as follows: Find $(\lambda, u) \in \mathcal{R} \times V$ such that

$$a(u, v) \ = \ \lambda b(u, v), \quad \forall v \in V, \tag{2}$$

where $V := H_0^1(\Omega)$ and

$$a(u, v) = \int_\Omega \big(\nabla u \cdot \nabla v + uv\big) d\Omega, \qquad b(u, v) = \int_\Omega uv d\Omega. \tag{3}$$

The norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are defined as

$$\|v\|_a = \sqrt{a(v, v)} \qquad \text{and} \qquad \|v\|_b = \sqrt{b(v, v)}.$$

It is well known that problem (2) has an eigenvalue sequence $\{\lambda_j\}$ (cf. [5, 12]):

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = 0$ when $i \neq j$. The first eigenvalue λ_1 is simple and in the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity.