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FULLY DIAGONALIZED CHEBYSHEV SPECTRAL METHODS FOR SECOND AND FOURTH ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. Fully diagonalized Chebyshev spectral methods for solving second and fourth order elliptic boundary value problems are proposed. They are based on appropriate base functions for the Galerkin formulations which are complete and biorthogonal with respect to certain Sobolev inner product. The suggested base functions lead to diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series. Numerical results demonstrate the effectiveness and the spectral accuracy.

Key words. Spectral method, biorthogonal Chebyshev polynomials, elliptic boundary value problems, numerical results.

1. Introduction

Chebyshev spectral methods for solving ordinary/partial differential equations on bounded domains have gained a rapid development during the last few decades, due to the Fast Fourier Transforms (FFT) for Chebyshev polynomials, see [1, 2, 3, 5, 7, 8, 9, 10, 11, 14, 17, 18]. The approximations for the general second and fourth order equations with constant coefficients (see for instance (19) and (31) below) also achieve the optimal convergence rates. However, as pointed out in [16], it is very important to choose an appropriate basis such that the resulting linear system is as simple as possible.

For the second order equation (19), one usually chose the basis in the early years as (cf. [6])

$$V_N = \mathrm{span}\{\phi_2(x), \phi_3(x), \cdots, \phi_N(x)\},$$

where

$$\phi_k(x) = \begin{cases} T_k(x) - T_0(x), & k \text{ even}, \\ T_k(x) - T_1(x), & k \text{ odd}, \end{cases}$$

with $T_k(x)$ being the kth degree Chebyshev polynomial. Unfortunately this basis leads to a linear system with full matrix and hence its usage is virtually prohibited in practice (see [16]). To this end, Shen [16] presented a new basis by choosing $\phi_k(x) = T_k(x) - T_{k+2}(x)$. Note that

$$-(\phi_j'', \phi_k)_\omega = \begin{cases} 2\pi(k+1)(k+2), & j = k, \\ 4\pi(k+1), & j = k+2, k+4, k+6, \cdots, \\ 0, & j > k \text{ or } j+k \text{ odd}, \end{cases}$$

where $\omega(x)$ is the Chebyshev weight function. Hence the matrices of the resulting linear systems are sparse and possess special structures. For the fourth order

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equation (31), Shen [16] also proposed a new basis

$$\psi_k(x) = T_k(x) - \frac{2(k+2)}{k+3}T_{k+2}(x) + \frac{k+1}{k+3}T_{k+4}(x), \quad 0 \le k \le N-4.$$

The matrix with the term $(\psi_j'', (\psi_k\omega)'')$ in the resulting linear system is not sparse, but still possesses special structures. Benefiting from these special matrix structures, Shen [16] further derive some efficient algorithms. However, in many cases, people still want to obtain a set of Fourier-like basis functions (see [4, 15]), which are orthogonal to each other with respect to certain Sobolev inner product involving derivatives, and thus the corresponding algebraic system is diagonal (see [19]).

Recently, Liu, Li and Wang [12, 13] constructed the Fourier-like Sobolev orthogonal basis functions based on generalized Laguerre functions, and applied them to the Dirichlet and Robin boundary value problems of second and fourth order elliptic equations on the half line. The numerical experiments indicate the suggested algorithms in [12, 13] are simple, fast and stable, and possess high accuracy.

Motivated by [12, 13, 19], the main purpose of this paper is to construct the Fourier-like basis functions for Chebyshev-Galerkin spectral methods of elliptic boundary value problems on bounded domain. Since the Chebyshev weight function will destroy the symmetry in the weak form of differential equations, we cannot design the basis functions which are mutually orthogonal with respect to the Sobolev inner product. Alternatively, we shall construct two kinds of basis functions which are biorthogonal with respect to the Sobolev inner product originated from the coercive bilinear form of the elliptic equation. For this purpose, we first design four kinds of special polynomials composed of Chebyshev polynomials, from which we further derive the basis functions for fully diagonalized Chebyshev-Galerkin spectral methods, which are biorthogonal with respect to the Sobolev inner product. Then stable and efficient algorithms are proposed for second and fourth order Dirichlet boundary value problems. Particularly, both the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier series, respectively.

The remainder of the paper is organized as follows. In Section 2, we first make conventions on the frequently used notations, and then design four kinds of special polynomials and introduce their basic properties. In Section 3, we construct the biorthogonal basis functions with respect to the Sobolev inner product associated with the second order Dirichlet boundary value problems, and present some numerical results. Section 4 is then devoted to the implementation of the fully diagonalized Chebyshev-Galerkin spectral methods for the fourth order Dirichlet boundary value problems. The final section is for some concluding remarks.

2. Chebyshev polynomials

2.1. Notations and preliminaries. Let $I = (-1, 1)$ and $\chi(x)$ be a weight function. Define

$$L^2_\chi(I) = \{v \mid v \text{ is measurable on } I \text{ and } \|v\|_\chi < \infty\},$$

with the following inner product and norm,

$$(u, v)_\chi = \int_I u(x)v(x)\chi(x)dx, \quad \|v\|_\chi = (v, v)_\chi^{\frac{1}{2}}, \quad \forall u, v \in L^2_\chi(I).$$

For simplicity, we denote $\frac{d^k v}{dx^k} = v^{(k)}$, $\frac{d^2 v}{dx^2} = v''$ and $\frac{dv}{dx} = v'$. For any integer $m \ge 0$, we define

$$H^m_\chi(I) = \{v \mid v^{(k)} \in L^2_\chi(I),\ 0 \le k \le m\},$$