INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 1-2, Pages 26–47

THE h-p VERSION OF THE CONTINUOUS PETROV-GALERKIN METHOD FOR NONLINEAR VOLTERRA FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH VANISHING DELAYS

LIJUN YI[1] AND BENQI GUO[1,2,*]

In memory of Professor Ben-yu Guo

Abstract. We investigate an *h-p* version of the continuous Petrov-Galerkin method for the nonlinear Volterra functional integro-differential equations with vanishing delays. We derive *h-p* version a priori error estimates in the L^2-, H^1- and L^∞-norms, which are completely explicit in the local discretization and regularity parameters. Numerical computations supporting the theoretical results are also presented.

Key words. *h-p* version, continuous Petrov-Galerkin method, nonlinear Volterra functional integro-differential equations, vanishing delays.

1. Introduction

We study the numerical solutions for the nonlinear Volterra functional integro-differential equation (VFIDE) with vanishing delays:

$$\begin{cases} u'(t) = f(t, u(t), u(\theta(t))) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), & t \in I := [0, T], \\ u(0) = u_0, \end{cases} \tag{1}$$

corresponding to the Volterra integral operators

$$(\mathcal{V}u)(t) := \int_0^t K_1(t,s) G_1(s, u(s)) ds, \qquad (\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_2(t,s) G_2(s, u(s)) ds,$$

where the delay function θ is subject to the following conditions:

(C1) $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$,

(C2) $\theta'(t) \geq q_0 > 0$ for all $t \in I$.

We assume that f and G_i with $i = 1, 2$ are given functions. Moreover, the kernels $K_1(t,s)$ and $K_2(t,s)$ are continuous on $D := \{(t,s) : 0 \leq s \leq t,\ t \in I\}$ and $D_\theta := \{(t,s) : 0 \leq s \leq \theta(t),\ t \in I\}$, respectively.

During the past few decades, many numerical methods have been proposed and analyzed for the VFIDEs. Among those a large number of methods are based on the h-version approach, which means that the convergence is achieved by decreasing the size of time steps at a fixed and typically low approximation order. For an overview of the lower-order methods developed for the VFIDEs, the reader can refer to monographs [3, 5] and the references therein. In contrast, the higher-order methods, for example, the p- and h-p version methods employ (varying) high order approximation polynomials. Particulary, the h-p version method allows for locally varying time steps and approximation orders, which can significantly enhance the numerical accuracy. The h-p version continuous and discontinuous

Received by the editors February 8, 2017 and, in revised form, May 19, 2017.
2000 *Mathematics Subject Classification.* 65L60, 65R20, 65L70, 41A10.
*Corresponding author.

Galerkin methods were introduced for initial-value problems in [9, 17, 19], for delay differential equations in [6], for parabolic problems in [10], and for Volterra integro-differential equations in [4, 8, 18, 20]. Moreover, some other high-order methods, such as the spectral methods were developed for various Volterra integro-differential equations with delays; see, e.g., [1, 13, 14, 15, 16, 21]. However, to the best of our knowledge, there is no work that considers the h-p version Galerkin method for nonlinear VFIDEs.

The purpose of the current work is to present and analyze an h-p version of the continuous Petrov-Galerkin (CPG) discretization scheme for the numerical approximation of the VFIDE (1) with vanishing delays. The Petrov-Galerkin method allows the trial and test spaces to be different, and it has become powerful tools for solving many kinds of differential equations (see e.g., [7, 12]). The CPG method presented in this paper is a hybrid of the continuous and discontinuous Galerkin methods with respect to time. More precisely, one uses continuous and piecewise polynomials for the trial spaces, but uses discontinuous and piecewise polynomials for the test spaces. With such choice of the trial and test spaces, we show that the CPG scheme defines a unique approximate solution, provided that a certain condition on the time steps is satisfied (which is completely independent of the approximation orders). We also describe in detail our implementation for the CPG scheme according to certain relationship between the delay function $\theta(t)$ and nodal points of the time partition. Moreover, we derive h-p version a priori error estimates that are completely explicit with respect to the local time steps, the local approximation orders, and the local regularity properties of the exact solution.

The remainder of this paper is organized as follows. In Section 2, we introduce the h-p version of the CPG method for the VFIDE (1) and prove existence and uniqueness of approximate solutions. We also give a detailed description of the computational form of the CPG scheme. In Section 3, we carry out a complete h-p version error analysis of the CPG method. In Section 4, we present some numerical experiments to verify the theoretical results. We end the paper with a summary and discussion in Section 5.

2. The h-p version of continuous Petrov-Galerkin method

In this section, we first introduce the h-p version of the CPG method for the VFIDE (1). We then show the existence and uniqueness of the approximate solutions. Finally, we discuss the numerical implementation of the CPG scheme.

2.1. Continuous Petrov-Galerkin discretization. Let $\mathcal{T}_h$ be a partition of the time interval I given by the points

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T.$$

We set $I_n = (t_{n-1}, t_n)$ and $k_n = t_n - t_{n-1}$ for $1 \leq n \leq N$. Let $k = \max_{1\leq n\leq N}\{k_n\}$. Moreover, we assign to each time interval I_n an approximation order $r_n \geq 1$ and introduce the degree vector $\mathbf{r} = \{r_n\}_{n=1}^N$. Then, the tuple $(\mathcal{T}_h, \mathbf{r})$ is called an h-p discretization of I. Next, we introduce the h-p version trial and test spaces

$$S^{\mathbf{r},1}(\mathcal{T}_h) = \{u \in H^1(I) : u|_{I_n} \in P_{r_n}(I_n), 1 \leq n \leq N\}$$

and

$$S^{\mathbf{r}-1,0}(\mathcal{T}_h) = \{u \in L^2(I) : u|_{I_n} \in P_{r_n-1}(I_n), 1 \leq n \leq N\},$$

respectively, where $P_{r_n}(I_n)$ denotes the space of polynomials of degree at most r_n on I_n.