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A QUADRILATERAL 'MINI' FINITE ELEMENT FOR THE STOKES PROBLEM USING A SINGLE BUBBLE FUNCTION

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Abstract. We consider a quadrilateral 'mini' finite element for approximating the solution of Stokes equations using a quadrilateral mesh. We use the standard bilinear finite element space enriched with element-wise defined bubble functions for the velocity and the standard bilinear finite element space for the pressure space. With a simple modification of the standard bubble function we show that a single bubble function is sufficient to ensure the inf-sup condition. We have thus improved an earlier result on the quadrilateral 'mini' element, where more than one bubble function are used to get the stability.

Key words. Stokes equations, mixed finite elements, Mini finite element, inf-sup condition, bubble function.

1. Introduction

A very simple finite element method for the Stokes problem for a simplicial mesh is presented by Arnold, Brezzi and Frotin [1], where the velocity space is discretised by using the standard linear finite element space enriched with element-wise bubble functions and the pressure space is discretised by using the standard linear finite element space. The enrichment of the velocity space is done to ensure the stability of the finite element method, and this increases one vector degree of freedom per element. An extension of the finite element method to the quadrilateral mesh is done by Bai [2], where the author enriches the velocity space with more than a single vector bubble function per element. The inf-sup condition is proved by using a macro element technique [9], where a single quadrilateral element is used as a macro element.

In this article we show that with a small modification of the standard bubble function we can get the stability just by using a single vector bubble function per element. The main difference with the technique proposed by Bai [2] is that it is not possible to show the inf-sup condition using a single quadrilateral element as a macro element. We need to use a macro element consisting of four quadrilateral elements to prove the inf-sup condition in our situation. Another relevant finite element method is presented by Lamichhane [8], where two different meshes are used to discretise the velocity and the pressure space, and a single vector bubble degree of freedom per element is used to get the stability. The pressure space is discretised by the space of piecewise constant functions on the dual mesh. However, the main difficulty of the technique presented by Lamichhane [8] is that the bubble function is obtained by multiplying the standard bubble function by the gradient of a bilinear basis function, and hence the bubble function cannot be defined on a reference element. The standard bubble function on the unit square is the lowest degree polynomial which vanishes on the boundary of the square. Here we modify the standard bubble function [1, 2] to get stability of the numerical scheme by using

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a single vector bubble function per element with a continuous pressure approximation. We also investigate two choices of bubble functions, where both of them can be defined on a reference element. Thus the main contribution of the paper is to introduce a modification of the standard bubble function so that the discrete velocity space can be enriched by a single bubble function per element to satisfy the inf-sup condition for a quadrilateral mesh. The idea can easily be extended to the three-dimensional case.

2. Stokes equations

This section is devoted to the introduction of the boundary value problem of the Stokes equations. Let Ω in $\mathbb{R}^2$, be a bounded domain with polygonal boundary Γ. For a prescribed body force $\boldsymbol{f} \in [L^2(\Omega)]^2$, the Stokes equations with homogeneous Dirichlet boundary condition in Γ reads

$$\begin{array}{lcll} -\nu\Delta \boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in} \quad \Omega \\ \operatorname{div} \boldsymbol{u} & = & 0 & \text{in} \quad \Omega \end{array} \tag{1}$$

with $\boldsymbol{u} = \boldsymbol{0}$ on Γ, where $\boldsymbol{u}$ is the velocity, p is the pressure, and ν denotes the viscosity of the fluid.

Here we use standard notations $L^2(\Omega)$, $H^1(\Omega)$ and $H^1_0(\Omega)$ for Sobolev spaces, see [4, 6] for details. Let $\boldsymbol{V} := [H^1_0(\Omega)]^2$ be the vector Sobolev space with inner product $(\cdot,\cdot)_{1,\Omega}$ and norm $\|\cdot\|_{1,\Omega}$ defined in the standard way: $(\boldsymbol{u},\boldsymbol{v})_{1,\Omega} := \sum_{i=1}^2 (u_i, v_i)_{1,\Omega}$, and the norm being induced by this inner product. We also define another subspace M of $L^2(\Omega)$ as

$$P = \left\{ q \in L^2(\Omega) : \int_\Omega q\, dx = 0 \right\}.$$

The weak formulation of the Stokes equations is to find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P$ such that

$$\begin{array}{llll} \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dx & + \int_\Omega \operatorname{div} v\, p\, dx & = \ell(\boldsymbol{v}), & \boldsymbol{v} \in \boldsymbol{V}, \\ \int_\Omega \operatorname{div} \boldsymbol{u}\, q\, dx & & = 0, & q \in P, \end{array} \tag{2}$$

where $\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx$. It is well-known that the weak formulation of the Stokes problem is well-posed [7]. In fact, if the domain Ω is convex, and $\boldsymbol{f} \in [L^2(\Omega)]^2$, we have $\boldsymbol{u} \in [H^2(\Omega)]^2$, $p \in H^1(\Omega)$ and the a priori estimate holds

$$\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \le C \|\boldsymbol{f}\|_{0,\Omega},$$

where the constant C depends on the domain Ω.

3. Finite element discretizations

We consider a quasi-uniform triangulation $\mathcal{T}_h$ of the polygonal domain Ω, where $\mathcal{T}_h$ consists of parallelograms. The finite element meshes are defined by maps from the reference square $\hat{K} = (0,1)^2$ to the actual parallelogram $K \in \mathcal{T}_h$. Let $\mathcal{Q}_1(\hat{K})$ be the space of bilinear polynomials in $\hat{K}$. We start with the finite element space of continuous functions whose restrictions to an element K are obtained by maps of *bilinear* functions from the reference element:

$$S_h := \left\{ v_h \in H^1_0(\Omega),\ v_h|_K = \hat{v}_h \circ F_K^{-1}, \hat{v}_h \in \mathcal{Q}_1(\hat{K}),\ K \in \mathcal{T}_h \right\}, \tag{3}$$

where $F_K : \hat{K} \to K$ is an affine mapping.

Remark 1. *For a convex quadrilateral the mapping $F_K : \hat{K} \to K$ is an isoparametric map, which may not be an affine mapping. The mapping becomes an affine mapping if K is a parallelogram.*