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A SIMPLE FAST ALGORITHM FOR MINIMIZATION OF THE ELASTICA ENERGY COMBINING BINARY AND LEVEL SET REPRESENTATIONS

XUE-CHENG TAI AND JINMING DUAN

Abstract. For curves or general interfaces, Euler's elastica energy has a wide range of applications in computer vision and image processing. It is however difficult to minimize the functionals related to the elastica energy due to its non-convexity, nonlinearity and higher order with derivatives. In this paper, we propose a very simple way to combine level set and binary representations for interfaces and then use a fast algorithm to minimize the functionals involving the elastica energy. The proposed algorithm essentially just needs to solve a total variation type minimization problem and a re-distance problem. Nowadays, there are many fast algorithms to solve these two problems and thus the overall efficiency of the proposed algorithm is very high. We then apply the new Euler's elastica minimization algorithm to image segmentation, image inpainting and illusory shape reconstruction problems. Extensive experimental results are finally conducted to validate the effectiveness of the proposed algorithm.

Key words. Euler's elastica energy, image segmentation, image inpainting, illusory shape, corner fusion, level set method, binary level set method, fast sweeping.

1. Introduction

For a two-dimensional curve γ, its elastica energy is defined as

$$E(\gamma) = \int_{\gamma} \left(a + b\kappa^2\right) ds. \tag{1}$$

Here κ is the curvature of the curve γ, ds is arc length and a and b are two positive parameters. If we set $b = 0$, $E(\gamma)$ measures the total length of the curve. If $a = 0$, then $E(\gamma)$ measures the twisting energy of the curve which is related to the curvature. The elastica energy has no difficulty to be extended for higher dimensional interface problems. For a function u defined on the domain Ω, the Euler's elastica energy of all level curves of u over Ω can be expressed as a functional of u by

$$E(u) = \int_{\Omega} \left(a + b\left|\nabla \cdot \frac{\nabla u}{|\nabla u|}\right|^2 \right) |\nabla u| \, dx. \tag{2}$$

In the field of image processing, Euler's elastica energy was first introduced by Nitzberg, Mumford, and Shiota for segmenting an image into objects with different depths in the scene [1]. Since then, it has been adapted to many fundamental problems in mathematical imaging. This includes image inpainting [2, 3, 4], image restoration [5, 6, 4, 7], image zooming [4] and image segmentation [8, 9, 10].

It is however nontrivial to minimize the functional (2) directly, because it involves non-differentiable, nonlinear and higher order terms. Recently, a lot of research have focused on the developments of fast and reliable numerical methods for minimizing curvature based functionals, including the multigrid algorithm [11], the homotopy method [12], augmented Lagrangian method (ALM) based algorithms [13, 14, 4],

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graph cut based algorithms [7, 15] and convex relaxation approaches [16, 17, 18, 19]. Among them the ALM based algorithms are particularly of interest, because the resulting minimization problems by the ALM can be implemented very easily and efficiently. ALM thus has become a powerful tool for developing efficient numerical schemes to deal with many nonlinear image processing models, such as the non-differentiable Rudin-Osher-Fatemi (ROF) model [20], the Euler's elastica and mean curvature models [4, 9, 14, 21]. The main idea of the ALM is to convert the original problem into a few subproblems, each of which is a very simple problem and can thus be solved efficiently. The minimizer of the original problem is obtained when the overall algorithm has converged. To minimize the Euler's elastica energy (2) with a data fidelity term $\mathcal{D}(u)$ using the ALM, (2) is first transformed into the following equivalent constrained minimization problem

$$\min_{u,\boldsymbol{p},\boldsymbol{m},\boldsymbol{n}} \int_\Omega \left(a + b(\nabla\cdot\boldsymbol{n})^2\right)|\boldsymbol{p}|\,dx + \mathcal{D}(u) \;\; s.t.\; \boldsymbol{p} = \nabla u,\; \boldsymbol{n} = \tfrac{\boldsymbol{p}}{|\boldsymbol{p}|}. \tag{3}$$

The constraint $\boldsymbol{n} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$ in (3) can be converted to

$$|\boldsymbol{m}| \leq 1, \quad |\boldsymbol{p}| = \boldsymbol{m}\cdot\boldsymbol{p}, \quad \boldsymbol{m} = \boldsymbol{n}.$$

With these new constraints, the augmented Lagrangian functional for (3) is:

$$\begin{aligned} E\left(u,\boldsymbol{p},\boldsymbol{m},\boldsymbol{n};\lambda_1,\boldsymbol{\lambda_2},\boldsymbol{\lambda_3}\right) = & \int_\Omega \left(a + b(\nabla\cdot\boldsymbol{n})^2\right)|\boldsymbol{p}|\,dx + \mathcal{D}(u) \\ & + \mu_1 \int_\Omega (|\boldsymbol{p}| - \boldsymbol{p}\cdot\boldsymbol{m})dx + \int_\Omega \lambda_1\left(|\boldsymbol{p}| - \boldsymbol{p}\cdot\boldsymbol{m}\right)dx \\ & + \frac{\mu_2}{2}\int_\Omega |\boldsymbol{p} - \nabla u|^2 dx + \int_\Omega \boldsymbol{\lambda_2}\cdot(\boldsymbol{p} - \nabla u)dx \\ & + \frac{\mu_3}{2}\int_\Omega |\boldsymbol{n} - \boldsymbol{m}|^2 dx + \int_\Omega \boldsymbol{\lambda_3}\cdot(\boldsymbol{n} - \boldsymbol{m})dx + \delta_{\mathcal{R}}(\boldsymbol{m}), \end{aligned} \tag{4}$$

where $\mathcal{R} = \left\{m \in L^2(\Omega) : |m| \leq 1 \text{ a.e. in } \Omega\right\}$ and $\delta_{\mathcal{R}}(\boldsymbol{m})$ is the characteristic function on the convex set $\mathcal{R}$, which is given by

$$\delta_{\mathcal{R}}(\boldsymbol{m}) = \begin{cases} 0 & \text{if } \boldsymbol{m} \in \mathcal{R} \\ +\infty & \text{otherwise} \end{cases}.$$

Moreover, μ_1, μ_2 and μ_3 are positive penalty parameters while λ_1, $\boldsymbol{\lambda_2}$ and $\boldsymbol{\lambda_3}$ are Lagrange multipliers. Since $\boldsymbol{m}$ is forced to be inside $\mathcal{R}$, $|\boldsymbol{m}| \leq 1$, $|\boldsymbol{p}| - \boldsymbol{m}\cdot\boldsymbol{p} \geq 0$ for any $\boldsymbol{p}$, and $|\boldsymbol{p}| - \boldsymbol{m}\cdot\boldsymbol{p} = 0$ if and only if $\boldsymbol{m} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$. This simplifies $\boldsymbol{p}$'s subproblem because quadratic term is avoided. The unknown $\boldsymbol{m}$ is introduced to decouple $\boldsymbol{p}$ and $\boldsymbol{n}$ such that $\boldsymbol{p}$'s subproblem can be solved by the shrinkage and $\boldsymbol{m}$'s subproblem by the fast Fourier transform. Notice that the fidelity term $\mathcal{D}(u)$ should be addressed properly according to different applications. For example, for noise removal with Gaussian noise, it is common to choose $\mathcal{D}(u) = \int_\Omega (u-f)^2 dx$. In such a case, the algorithm can be used directly and give fast numerical implementations. However, an additional variable should be introduced for the algorithm when $\mathcal{D}(u) = \int_\Omega |u-f|dx$, which is common for impulsive noise removal. We refer the reader to [4] for more details on the application of the ALM to different $\mathcal{D}(u)$. One now needs to minimize the augmented Lagrangian functional for each of the variables $u, \boldsymbol{p}, \boldsymbol{m}, \boldsymbol{n}$ by fixing the others. After all the variables are solved, the Lagrange multipliers $\lambda_1, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}$ should be updated. The procedure is repeated until all the variables have converged.

By considering the underlying relation between the length term and the curvature term in the Euler's elastica energy, in this paper we propose a novel algorithm for