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ANALYSIS OF AN EMBEDDED DISCONTINUOUS GALERKIN METHOD WITH IMPLICIT-EXPLICIT TIME-MARCHING FOR CONVECTION-DIFFUSION PROBLEMS

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Abstract. In this paper, we analyze implicit-explicit (IMEX) Runge-Kutta (RK) time discretization methods for solving linear convection-diffusion equations. The diffusion operator is treated implicitly via the embedded discontinuous Galerkin (EDG) method and the convection operator explicitly via the upwinding discontinuous Galerkin method.

Key words. Embedded discontinuous Galerkin method, upwinding discontinuous Galerkin method, implicit-explicit Runge-Kutta time-marching scheme, convection-diffusion equation, stability, error estimate, energy method.

1. Introduction

In this paper, we propose and analyze an implicit-explicit embedded discontinuous Galerkin (IMEX-EDG) method for solving the linear convection diffusion equation. We use the IMEX Runge-Kutta time discretization [1] that treats the diffusion term implicitly via the embedded discontinuous Galerkin (EDG) method [7, 6] and the convection term explicitly via the upwinding discontinuous Galerkin method [9]. For a detailed discussion on IMEX RK schemes, see [1, 3, 8] and references therein.

The EDG methods, originally introduced for linear shells in [7], is obtained from hybridizable discontinuous Galerkin (HDG) methods [5] by simply reducing the space of the hybrid (interface) unknowns by requiring them to be continuous across the mesh skeleton. It reduces the globally coupled degrees of freedom (after hybridization) to exactly those for a continuous Galerkin formulation (after static condensation).

Here we consider three specific Runge-Kutta type IMEX schemes given in [1] from first to third order accuracy. Coupling with the EDG (diffusion) and upwinding DG (convection) spatial discretization, we give the stability analysis and error estimates by the energy method. Our work is inspired from [10, 11, 12], where the authors analyzed IMEX time stepping coupled with local discontinuous Galerkin (LDG) methods for linear and nonlinear convection diffusion equations. The only difference of the IMEX-LDG and IMEX-EDG methods is on the discretization of the diffusion operator. While the theoretical results are similar for both spatial approaches, the IMEX-EDG methods is more computationally efficient due to a smaller number of globally coupled degrees of freedom. On a fixed triangular mesh in two dimensions, using polynomials of degree k approximations, the LDG method results a globally coupled linear system of size $N_t(k+1)(k+2)/2 \approx N_v(k+1)(k+2)$, while the EDG method results a globally coupled linear system of size $N_v + N_e(k-1) \approx N_v(3k-2)$. Here N_v, N_e, and N_t

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are the numbers of vertices, edges, and triangles. We remark that while we can equally use the HDG methods [5] (with a discontinuous hybrid space) to discretize the diffusion operator, the EDG methods is more efficient in terms of the number of globally coupled degrees of freedom.

The paper is organized as follows. In Section 2 we present the spatial discretization for the model convection diffusion problem and give some preliminary results. Then, in Section 3, we present and analyze the fully discrete schemes with IMEX RK time discretization. Several numerical tests are presented in Section 4 to verify the main results in Section 3. Finally, we conclude in Section 5.

2. Semi-discretization with EDG for diffusion and upwinding DG for convection

In this section, we present the spatial discretization for the following linear convection-diffusion problem:

$$u_t + \nabla\cdot(\boldsymbol{\beta} u) - \nabla\cdot(\epsilon\,\nabla u) = 0, \qquad (\boldsymbol{x}, t) \in \Omega_T = \Omega\times(0,T], \tag{1a}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega, \tag{1b}$$

with a periodic boundary condition. Here $\Omega \in \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded rectangular domain, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$ is a constant velocity field, ϵ is the diffusion coefficient, and $u_0(\boldsymbol{x})$ is the initial solution.

We use the EDG scheme [7, 6] to discretize the diffusion operator and the upwinding DG scheme [9] to discretize the convection operator. We present properties of these schemes that will be used for the analysis of the fully discrete schemes in Section 3.

We first collect some notation that will be used throughout the paper.

2.1. Notation and preliminaries. We denote by $\|\cdot\|_{H^m(D)}$ the standard H^m-Sobolev norm on the domain $D \subset \mathbb{R}^d$. When $m = 0$, we simplify the notation and denote by $\|\cdot\|_D$ the L^2-norm on D.

We denote by $\mathcal{T}_h := \{K\}$ a quasi-uniform, shape-regular conforming simplicial triangulation of Ω, and by $\mathcal{E}_h$ the mesh skeleton consists the set of facets F (element nodes in $1d$, edges in $2d$, and faces in $3d$) of the simplicial elements $K \in \mathcal{T}_h$. We denote by ∂K the element boundary of an element K.

We denote by Volume(K) and Volume(∂K) the volume and surface area of K in $3d$. In $2d$, Volume(K) is the area of the triangle K, and Volume(∂K) is the perimeter length. And in $1d$, Volume(K) is the length of the interval K, and Volume(∂K) is set to be 2. We set $h_K := \mathrm{diam}(K)$ and $h := \max_{K\in\mathcal{T}_h} h_K$.

Associated with the triangulation and mesh skeleton, we define the discontinuous (cell-wise) finite element spaces (on $\mathcal{T}_h$) and continuous (facet-wise) finite element space (on $\mathcal{E}_h$):

$$\boldsymbol{R}_h := \{\boldsymbol{r} \in L^2(\mathcal{T}_h)^d : \boldsymbol{r}|_K \in \mathcal{P}^d_{k-1}(K), K \in \mathcal{T}_h\}, \tag{2a}$$

$$V_h := \{v \in L^2(\mathcal{T}_h) : \ v|_K \in \mathcal{P}_k(K), \quad K \in \mathcal{T}_h\}, \tag{2b}$$

$$M_h := \{\widehat{v}_h \in C^0(\mathcal{E}_h) : \widehat{v}_h|_F \in \mathcal{P}_k(F), \quad F \in \mathcal{E}_h\}, \tag{2c}$$

for $k \geq 1$. Here $\mathcal{P}_m(K)$ ($\mathcal{P}^d_m(K)$) stands for the space of scalar (vector) polynomials of degree at most m. We use the convention that $\mathcal{P}_m(F)$ is the space of constants