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# NONCONFORMING FINITE VOLUME METHODS FOR SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

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**Abstract.** This paper is devoted to analyze of nonconforming finite volume methods (FVMs), whose trial spaces are chosen as the nonconforming finite element (FE) spaces, for solving the second order elliptic boundary value problems. We formulate the nonconforming FVMs as special types of Petrov-Galerkin methods and develop a general convergence theorem, which serves as a guide for the analysis of the nonconforming FVMs. As special examples, we shall present the triangulation based Crouzeix-Raviart (C-R) FVM as well as the rectangle mesh based hybrid Wilson FVM. Their optimal error estimates in the mesh dependent $H^1$-norm will be obtained under the condition that the primary mesh is regular. For the hybrid Wilson FVM, we prove that it enjoys the same optimal error order in the $L^2$-norm as that of the Wilson FEM. Numerical experiments are also presented to confirm the theoretical results.

**Key words.** Nonconforming finite volume method, elliptic boundary value problems.

## 1. Introduction

T $T$ Preserving certain local conservation laws and flexible algorithm constructions are the most attractive advantages of the FVM. Due to its strengths, the FVM has been widely used in numerical solutions of PDEs, especially in computational fluid dynamics, computational mechanics and hyperbolic problems (cf. [13, 20, 26]). In the past several decades, many researchers have studied this method extensively and obtained some important results. We refer to [2, 4, 7, 8, 17, 22, 27] for an incomplete list of references.

Most of the existing work about FVMs for solving the second order elliptic boundary value problems focuses on the conforming schemes, which employ the standard conforming FE spaces as their trial spaces, see [1, 9, 14, 15, 21] for triangulation based FVMs and [23, 24, 28] for rectangle mesh based FVMs. There are little work about the nonconforming FVMs (cf. [3, 5, 6, 10]). A general construction of higher-order FVMs based on triangle meshes was proposed in a recent paper [9] for solving the second order elliptic boundary problems and a unified approach for analyzing the methods was developed. We feel it is necessary to establish a unified theoretical framework for the nonconforming FVMs for solving boundary value problems of the two dimensional elliptic equations.

In this paper, we shall establish a convergence theorem applicable to the nonconforming triangle mesh based FVMs as well as the rectangle mesh based FVMs for solving the second order elliptic boundary problems. We will see that comparing with the conforming FVMs, verifying the uniform boundedness and the uniform ellipticity of the family of the discrete bilinear forms is still a task for the nonconforming FVMs. Moreover, there is an additional nonconforming error to estimate.

As a special example, the C-R FVM will be presented in this paper, whose trial space is the C-R FE space with respect to the primary triangulation (cf. [12]) and test space is spanned by the characteristic functions of the control volumes in the dual partition. Based on the C-R element, paper [5] considered the FVM for

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solving elliptic boundary problems in 2-D and obtained the optimal order error estimates in the $L^2$-norm and a mesh dependent $H^1$-norm. The the reaction term of the elliptic equation there was not generalized by the Petrov-Galerkin formulation. Instead, this term was discretized using a diagonal matrix. By virtue of the same discretization skill of the reaction term, paper [3] considered the FVM based on the C-R element for the non-self-adjoint and indefinite elliptic problems and proved the existence, uniqueness and uniform convergence of the FV element approximations under minimal elliptic regularity assumption. In the nonconforming FVM schemes presented in this paper, we employ the generalization of the Petrov-Galerkin formulation to get the discrete bilinear forms. This will be beneficial to the development of a general framework for the numerical analysis of the methods. We will prove two discrete norm inequalities which lead to the uniform boundedness of the family of the discrete bilinear forms and we will establish the uniform ellipticity of the family of the discrete bilinear forms. We also show that the nonconforming error is equal to zero and in turn get the optimal error estimate in the mesh dependent $H^1$-norm for the C-R FVM.

Another special example, the hybrid Wilson FVM, will also be presented in this paper. The trial space of the hybrid Wilson FVM is the Wilson FE space with respect to the primary rectangle mesh and test space is panned by the characteristic functions of the control volumes combined with certain linearly independent functions of the trial spaces. The hybrid FVM was initially constructed for a triangulation based quadratic FVM in [7] and further studied in [9]. We will show that the convergence order of the hybrid Wilson FVM in the mesh dependent $H^1$-norm is $O(h)$, the same as that for the Wilson FE method (cf. [25]). The discrete bilinear form of the FVM is dependent on the meshes which introduce a major obstacle for the $L^2$-norm error estimate of the hybrid Wilson FVM. We note that the test space of the hybrid Wilson FVM is produced by the piecewise constant functions with respect to the dual partition and the nonconforming functions of the trial space. Then, we may borrow some useful techniques used for the $L^2$-error estimate of the lower-order FVM ([23]) and the Wilson FEM ([25]). We will verify that the convergence order of the hybrid Wilson FVM in the $L^2$-norm is $O(h^2)$, the same as that for the Wilson FE method.

The remainder of this paper is organized as follows. In section 2, we describe the framework of the nonconforming FVMs for the second order elliptic boundary value problems and develop a convergence theorem. Sections 3 and 4 are devoted to the discussion of the C-R FVM and the hybrid Wilson FVM respectively. Their discrete norm inequalities will be proved, nonconforming error term will be estimated and uniform ellipticity will be established. Then, their the optimal error estimates in the mesh dependent $H^1$-norm are derived, respectively. In section 5, we discuss the $L^2$-norm error estimate for the hybrid Wilson FVM for solving the Poisson equation. In the last section, we present a numerical example to confirm the convergence results in this paper.

In this paper, the notations of Sobolev spaces and associated norms are the same as those in [11] and $C$ will denote a generic positive constant independent of meshes and may be different at different occurrences.

## 2. The Nonconforming FVMs for Elliptic Equations

Let $\Omega$ be a polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Suppose that $\mathbf{a} := [a_{ij}(x)]$ is a $2 \times 2$ symmetric matrix of functions $a_{ij} \in W^{1,\infty}(\Omega)$ and $f \in L^2(\Omega)$ and that $b$ is a smooth, nonnegative and real function. We consider the Dirichlet problem of