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DIFFERENTIAL QUADRATURE-BASED SIMULATION OF A CLASS OF FUZZY DAMPED FRACTIONAL DYNAMICAL SYSTEMS

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Abstract. In this paper, a numerical approach for the simulation of a dynamical model with damping defined by the Riemann–Liouville fractional derivative and with uncertainty, that is fuzziness, is discussed. The proposed method exploits differential quadrature rules and a Picard–like recursion. The convergence is formally discussed. Some example applications, in the linear and nonlinear regime, confirm the theoretical achievements.

Key words. Fractional derivatives, differential quadrature rules, Picard-like approach, fuzzy sets.

1. Introduction

Dynamical systems with uncertainty intended as fuzziness have been deeply investigated (e.g. see [14, 31, 6, 25]). Fuzziness is a way to take into account an uncertainty which cannot be identified as randomness [42].

In this paper, the following second order fuzzy initial value problem in presence of a Riemann-Liouville fractional derivative is considered

$$\begin{aligned} L_t^{(2)}\tilde{u}(t) + \delta\ {}^{RL}D^{\beta}(\tilde{u}(t)) &= f(\tilde{u}(t)) + \tilde{g}(t), \\ L_t^{(i)}\tilde{u}(0) &= \tilde{a}_i, \qquad i = 0, 1 \end{aligned} \tag{1}$$

where $L_t^{(i)}$ is the ith-order derivative operator with respect to t, ${}^{RL}D^{\beta}$ is the Riemann–Liouville fractional derivative of order β, with $0 < \beta < 1$, $\delta \in \mathbb{R}$ a parameter. Besides, $f(\tilde{u}(t))$ is a functional form in $\tilde{u}$, $\tilde{g}(t)$ a given fuzzy-valued function and $\tilde{a}_i$ a fuzzy number, with $i = 0, 1$. Here, $\tilde{u}(t)$ represents the unknown fuzzy function for $t \in [0, T]$, with $T \in \mathbb{R}_+$. It should be pointed out that (1) may be seen as the fuzzification via the Zadeh's extension principle of the same problem but without fuzzy variables and parameters.

Broadly speaking, a fuzzy number can be seen as a more general representation of a real number, since it does not refer to a single value but to a set of possible values. From a computational perspective, a fuzzy quantity can be approximated by sets of closed intervals through the so–called α–cut approach. In a certain sense, this is congruent (even though different) with interval analysis (e.g. see [29]). Details about fuzzy numbers and all the relevant related issues will be provided in the next section. The problem corresponding to (1) in the non-fuzzy domain is relevant in structural dynamics, where fractional derivatives are mostly used for describing viscoelastic features of advanced materials [35]. Such kind of problems in the non-fuzzy domain has been mainly solved by means of perturbation techniques [35], which are known to involve demanding symbolic computations (e.g. [19]).

In this paper, fuzziness is considered in order to model an uncertain or imprecise system which is not a probabilistic dynamical system. Indeed, uncertainty in viscoelastic structures is a topic under discussion in the current literature [4].

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A numerical method is herein proposed to solve (1), by extending the approach presented in [37, 38] for solving linear integro–differential equations. More precisely, the proposed scheme foresees differential quadrature rules [9], which can be regarded as high-order finite-difference approximations [17, 28, 39, 40], and a Picard-like recursion into the fuzzy domain.

In the current literature, there are several examples of Picard-like approaches in the non-fuzzy domain (e.g. [15, 41, 34]), but not for problems like the one herein considered. Obviously, the method herein proposed reflects the transformation of the problem to be studied into a set of interval equations at α–levels. In particular, it follows some recent achievements in the field [7].

The approach herein proposed, in spite of its recursive nature, leads to a non–recursive approximate solution in the linear regime, by means of operational matrices and vectors of known quantities. The present scheme is different from the one proposed in [27], dealing with linear fuzzy partial differential equations. Convergence is herein formally discussed and some relevant example applications from the current literature, in the linear and nonlinear regime, are considered. Numerical results are in good agreement with the analytical solutions in the non-fuzzy domain, available in literature. The number of discrete points needed to get such solutions is small enough as well as the number of iterations for the nonlinear cases. Finally, it is worthwhile pointing out that the problem herein considered is different from the one in [13], where a linear system with a Caputo fractional derivative was numerically simulated by means of the homotopy perturbation method, without stating any property.

The paper is sectioned as follows. Section 2 introduces some basic notions. Section 3 presents the approach. In Section 4, some properties, and in particular convergence, are discussed. Section 5 is devoted to numerical experiments. The paper closes with some conluding remarks.

2. Preliminaries

In this section, some basic notions are provided (for more details, one can refer to classical textbooks, e.g. [30]). Throughout, $\mathbb{U}$ will denote a nonempty and closed set of $\mathbb{R}$.

Definition 2.1. *A* fuzzy number $\tilde{u}$ *is defined through a membership function* $\mu_u(x) : \mathbb{U} \to [0, 1]$ *and it satisfies* normality *and* convexity *on* $\mathbb{U}$.

The membership function maps each element from $\mathbb{U}$ to a membership value (or degree of membership), between a minimum, that is 0, and a maximum, that is 1, according to normality. Convexity requires that the membership function is piecewise continuous, but in compliance with normality. Arithmetic operations on fuzzy numbers can be approached either by means of the membership function, i.e. through the Zadeh extension principle, or by means of the α-cuts representation [21].

It should be pointed out that in a classical (or crisp) set, the membership function allows each element to have just 0 or 1 value, by meaning that a given element can belong or not to that set.

Definition 2.2. *An* α-cut *of the fuzzy number* $\tilde{u}$ *is the crisp set defined by*

$$[\tilde{u}]_\alpha = \{x \in \mathbb{U} : \mu_u(x) \geq \alpha\}, \qquad \alpha > 0. \tag{2}$$

Notice that for $\alpha = 0$, the α-cut of a fuzzy number $\tilde{u}$ is $[\tilde{u}]_0 = cl(\{x \in \mathbb{U} : \mu_u(x) > 0\})$, where cl denotes closure in the standard topology of $\mathbb{U}$.