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CONVERGENCE OF DISCONTINUOUS FINITE VOLUME DISCRETIZATIONS FOR A SEMILINEAR HYPERBOLIC OPTIMAL CONTROL PROBLEM

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Abstract. In this paper, we discuss discontinuous finite volume approximations of the distributed optimal control problems governed by a class of semilinear hyperbolic partial differential equations with control constraints. The spatial discretization of the state and costate variables follows discontinuous finite volume schemes with piecewise linear elements, whereas three different strategies are used for the control approximation: variational discretization, piecewise constant and piecewise linear discretization. As the resulting semi-discrete optimal system is non-symmetric, we have employed *optimize then discretize* approach to approximate the control problem. *A priori* error estimates for control, state and costate variables are derived in suitable natural norms. The present analysis is an extension of the analysis given in Kumar and Sandilya [Int. J. Numer. Anal. Model. (2016), 13: 545-568]. Numerical experiments are presented to illustrate the performance of the proposed scheme and to confirm the predicted accuracy of the theoretical convergence rates.

Key words. Semilinear hyperbolic optimal control problems, variational discretization, piecewise constant and piecewise linear discretization, discontinuous finite volume methods, *a priori* error estimates, numerical experiments.

1. Introduction

It is well known that optimization problems governed by partial differential equations introduced in [25] have many applications in the field of science and technology. In particular, the hyperbolic optimal control problems arise in medical applications, acoustic problems as noise suppression and for optimal control in linear elasticity (cf. [1, 7, 33]). Although abundant literature is available on finite element analysis for elliptic and parabolic optimal control problems (see, e.g., [8, 10, 29, 30, 32]), there is relatively less work on hyperbolic optimal control problems (see, e.g., [15, 16, 31]). Most of these articles deal with conforming piecewise linear finite element discretizations for state and costate variables and control is discretized using piecewise constant or linear polynomials, and the rate of convergence for control is of $\mathcal{O}(h)$ and $\mathcal{O}(h^{3/2})$ when piecewise constant and linear polynomials are used, respectively. In order to improve the order of convergence, Hinze proposed a variational discretization approach for optimal control problems with control constraints in which control set is not discretized explicitly but discretized by a projection of the discrete costate variables, for details see [14]. For this new scheme, it has been shown that the rate of convergence for the control is of $\mathcal{O}(h^2)$.

Due to local conservation properties and other attractive features, finite volume methods have been extensively used for the approximation of partial differential equations obeying some conservation laws. They can also be considered as Petrov-Galerkin methods in which the finite dimensional trial and test spaces consist piecewise linear polynomials and piecewise constant functions, respectively. For more details on finite volume methods, kindly see [6, 12, 13, 24] and references

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therein. Since the test space associated with the dual grid is piecewise constant, finite volume methods have some computational advantages over continuous finite element methods. Due to computational efficiency and simplicity, these methods are widely used for the approximation of linear elliptic, parabolic and hyperbolic optimal control problems (see e.g. [26, 27, 28]) and *a priori* error estimates have also been established. In these articles, variational discretization approach is used to approximate control variable and optimal order of convergence is obtained.

On the other hand, discontinuous Galerkin methods are very appealing to the scientific community because of their desirable properties like: mesh adaptivity, locally conservative, suitability for parallel computing, use of high order polynomials and no inter element continuity requirement (generally imposed on continuous and non-conforming finite element spaces) etc. A few contributions are available (cf. [11, 29, 30, 32]) which deal with discontinuous Galerkin methods for linear and semilinear parabolic optimal control problems. In order to utilize the desirable properties of both finite volume and discontinuous Galerkin methods, Ye in [35] proposed a hybrid scheme called discontinuous finite volume (DFV) methods to approximate linear elliptic problems. In DFV scheme, discontinuous piecewise linear functions are used in trial space whereas test space consists of piecewise constant functions. Later, with the appropriate modifications these methods have been applied to elliptic, parabolic and certain fluid flow problems (for details, see [3, 5, 18, 19, 20]). Recently, DFV methods have been applied to solve optimal control problems governed by elliptic [21], semilinear parabolic [22] and Brinkman [23] equations and here we extend these ideas to the case of semilinear hyperbolic optimal control problem.

For the numerical solution of optimal control problems, there are two different strategies- *optimize-then-discretize* and *discretize-then-optimize*. In *optimize-then-discretize* approach, the optimality conditions at the continuous level are formulated first and then one proceeds to the discretization step; whereas in *discretize-then-optimize* approach one first discretizes the continuous problem and then derives the optimality conditions accordingly. For non-symmetric discrete formulations, these two approaches need not coincide as they may lead to different discrete adjoint equations (see [2]). In general, finite volume element formulation is non-symmetric and the authors in [26, 27, 28] have employed *optimize-then-discretize* technique to discretize the optimal control problems. In the light of these articles and applicability of *optimize-then-discretize* approach, in this article, we will also undertake the same strategy (*optimize-then-discretize*) for the approximation of the concerned control problem.

The rest of this article is organized in the following manner. The remaining part of this section deals with some standard notations, statement of the governing problem and the corresponding optimality conditions. In Section 2, we apply DFV scheme to the considered optimal control problem and obtain its discrete formulation. Section 3 deals with *a priori* error estimates for different types of control discretization. In Section 4, we present numerical experiments to illustrate the theoretical results and performance of the method. Finally, based on theoretical and computational observations, some conclusions are drawn in Section 5.

Notations. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$ and T be a positive time that defines the time interval $I := (0, T]$. The standard notations are used for the Lebesgue spaces $L^p(\Omega)$ and the Sobolev Spaces $H^s(\Omega)$ and their associated norms $\|\cdot\|_{s,\Omega}$ and seminorms $|\cdot|_{s,\Omega}$. Also, we write $H^0(\Omega) := L^2(\Omega)$