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A SEMI DISCRETE MODEL FOR MORTGAGE VALUATION AND ITS COMPUTATION BY AN ADAPTIVE FINITE ELEMENT METHOD

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Abstract. In traditional models for valuation of mortgages with a stochastic interest rate, one parabolic equation starting from the maturity is assumed to govern the whole life of a mortgage. Following the valuation of zero-coupon bond, a new model is proposed, where an initial value problem is restarted after a mortgage payment each month. In addition, the low and high limits on the interest rate are incorporated into the initial-boundary value problems, so that the partial differential equation remains regular and the solution better approximates the real value. We show the existence and uniqueness of the solution and the free boundary (which determines early prepayment). A finite element method is introduced with a convergence analysis. Numerical tests are presented and the results are interpreted for guiding mortgage practice.

Key words. Finite element method, parabolic equation, free boundary problem, mortgage valuation.

1. Model

We consider the standard fixed rate mortgage, where the loan borrower pays an equal amount of money to the lender for the duration of the contract. Typically the borrower has a choice of early settlement, for example, by refinance if a much lower fixed-rate mortgage is available, or by the fund from his or her other investment where the return is too low. The free boundary computed by a mortgage model would help such a borrower to determine if and when to pay off a loan. On the other hand, the mortgage valuation would help financial institutions to assess their loan equity, for example, in issuing mortgage backed security or bond.

The mortgage securities constitute one of the world's largest fixed income market. An adequate and efficient model for pricing mortgage contracts is not only useful for bankers and home owners to make financial decisions, but also critical for the sustainable development of mortgage market. For this reason, there exist considerable literature dealing with mortgage valuation and relevant topics. Most researchers have studied the problem from theoretical option pricing viewpoint, where the mortgage contracts are treated as an American style financial option [9, 1, 4]. A survey in this regard can be found in [10]. Since financial option valuation rarely assumes closed form solutions, efforts have been made to solve such problems numerically. For instance, a binomial iteration scheme is proposed in [12]. A projected successive over-relaxation iterative method is applied in [14]. A Monte-Carlo simulation method is tried in [8]. One notices (see [17, 7, 16], for instance) that usual numerical techniques such as binomial method typically provide poor accuracy and stability, which are mainly attributed to the difficulty in handling free boundary conditions, in addition to low convergence rate. In this manuscript, we propose a new model simplifying the free boundary setting while reflecting the real market practice.

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Assuming the interest rate follows the CIR [3] brownian motion, a mortgage valuation $V(x,t)$ either grows at the loan rate or decays at the higher market interest rate, cf. [11],

$$\begin{aligned} &\max\Big\{\frac{\partial V}{\partial t}-\frac{\sigma^2}{2}x\frac{\partial^2 V}{\partial x^2}-k(\theta-x)\frac{\partial V}{\partial x}+xV-m, \\ (1)\qquad &\qquad V-\frac{m}{c}(1-e^{-ct})\Big\}=0, \quad -\infty<x<\infty,\ 0\le t\le T, \\ &\qquad V(x,0)=0. \end{aligned}$$

(Details and notations are provided next section.) This model is not well posed (may have multiple solutions) and is not computable. Assume there is only one free boundary ($h(t)$ one point separating PDE and ODE regions) and assume the two pieces of solution of (1) join smoothly, then the model is equivalent to an over-constrained system of differential equations, as commonly done in the American option models and in the zero-coupon bond models: Find $h(t)$ and $V(x,t)$ such that

$$(2)\qquad \begin{aligned} \frac{\partial V}{\partial t}&=\frac{\sigma^2}{2}x\frac{\partial^2 V}{\partial x^2}+k(\theta-x)\frac{\partial V}{\partial x}-xV+m, && h(t)<x<\infty,\ 0<t<T, \\ V(h(t),t)&=\frac{m}{c}(1-e^{-ct}), && 0<t<T, \\ \frac{\partial}{\partial x}V(h(t),t)&=0, && 0<t<T, \\ V(x,0)&=0, && x\ge c, \\ h(0)&=c. \end{aligned}$$

Like the solutions to the American option problems, or to the zero-coupon bond problems, the mortgage valuation model (2) is not well posed either. By the free-boundary condition $V_x(h(t),t)=0$, the PDE provides a solution $V(x,t)>V(h(t),t)$ for some $x>h(t)$. That is, a second free-boundary $\tilde h(t)(>h(t))$ would be created at which $V(\tilde h(t),t)=(m/c)(1-e^{-ct})$ and $V_x(\tilde h(t),t)<0$. This violates the refinance principle.

As both models are not well posed, we propose a new model where the free boundary is converted to an initial condition of one parabolic PDE: in $(x,t)\in(c_{\min},c_{\max})\times(0,\frac{1}{12})$

$$(3)\qquad \frac{\partial V^{(n)}}{\partial t}-\frac{\sigma^2}{2}x\frac{\partial^2 V^{(n)}}{\partial x^2}-k(\theta-x)\frac{\partial V^{(n)}}{\partial x}+xV^{(n)}=0,$$

for $n=1,2,\ldots$, with initial and boundary conditions

$$\begin{aligned} V^{(n)}(x,0)&=me^{-\frac{\max\{c,x\}}{12}}+\min\left\{V^{(n-1)}(x,\frac{1}{12}),V^{(n-1)}(c_{\min},0)e^{-\frac{c}{12}}\right\}, \\ V^{(n)}(c_{\min},t)&=V^{(n)}(c_{\min},0)e^{-c_{\min}t}, \\ V^{(n)}(c_{\max},t)&=V^{(n)}(c_{\max},0)e^{-c_{\max}t}, \\ V^{(0)}(x,0)&=0. \end{aligned}$$

(Details are given in the next section.) That is, we limit the time of refinance to the time of monthly mortgage payment. This way we avoid mathematical problems in the other two models and provide a practical and computable model. A significance of the new model is its avoidance of numerical computation of the exponential growth of the old model (1), by entering the exponential growth term as an exact initial condition. We will show the uniqueness and well-posedness of the model (3)