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A NOTE ON THE CONVERGENCE OF A CRANK-NICOLSON SCHEME FOR THE KDV EQUATION

RAJIB DUTTA AND NILS HENRIK RISEBRO

Abstract. The aim of this paper is to establish the convergence of a fully discrete Crank-Nicolson type Galerkin scheme for the Cauchy problem associated to the KdV equation. The convergence is achieved for initial data in L^2, and we show that the scheme converges strongly in $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}))$ to a weak solution for some $T>0$. Finally, the convergence is illustrated by a numerical example.

Key words. Crank-Nicolson scheme, KdV equation.

1. Introduction

In this paper we analyze a fully discrete Crank-Nicolson second order accurate scheme for the initial value problem associated to the KdV equation

$$\begin{cases} u_t + (\frac{u^2}{2})_x + u_{xxx} = 0, & x \in \mathbb{R}\times(0,T) \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $T>0$ is fixed, $u:\mathbb{R}\times[0,T)\to\mathbb{R}$ is the unknown, and u_0 is the initial data.

This equation originally arose as a model for shallow water waves, but it has later been used for models of varying phenomena, such as magneto-acoustic waves in plasmas, lattice waves etc. It has also been widely studied from the purely mathematical side, the delicate balance between nonlinear convection and dispersion allows for a rich family of explicit solutions called solitons. Solitons were originally discovered by Zabusky and Kruskal using numerical methods [17]. To obtain explicit, but complicated, formulas for solitons, one can use the Bäcklund transform. Solitons are localized, meaning that they tend rapidly to a constant for large $|x|$, and they interact in a particle like manner.

Despite the fact that solitons were discovered using numerical methods, it is quite difficult to approximate solutions to the KdV equations numerically. A numerical method must take into account both the nonlinear convection coming from the term uu_x and the (hard to compute) dispersive waves originating from u_{xxx}. When approximating smooth solutions, to the best our knowledge, spectral methods [9, 13, 11] or discontinuous Galerkin methods [16, 15, 3] most efficiently produce accurate approximations. These methods are essentially semi-discrete, where the time variable is kept as a continuous variable, and their fully discrete counterparts are hard to analyze, see however [9] in which a very efficient fully discrete version is presented.

Regarding fully discrete methods, a simple first order method (which is a discretization of the semi-discrete method used by Sjöberg to first give an existence proof for the Cauchy problem for the KdV equation [12]) is analyzed and shown to converge to a solution [7]. However in practice this method requires a very fine grid, and correspondingly large computational effort, to produce acceptable

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solutions. By using a higher order approximation in space and fully implicit time stepping [5], the efficiency improves slightly, while the resulting scheme is shown to converge for initial data in L^2. The purpose of this note is to analyze a second-order-in-time version of the scheme presented in [5], and to show that one still has convergence for general L^2 initial data, while in practice the scheme is second order accurate, and comparable with the second order discontinuous Galerkin scheme of [6].

We shall now briefly and informally explain our strategy. Define, *for the moment*, a weak solution to the KdV equation to be a function $u(t,x)$ such that $u \in C^1([0,\infty); H^2(\mathbb{R}))$ and that for all $v \in H^2(\mathbb{R})$,

$$(u_t, v) + (uu_x, v) + (u_x, v_{xx}) = 0, \tag{2}$$

where $(\cdot,\cdot)$ denotes the usual L^2 inner product. We propose a Crank-Nicolson discretization of this equation. Let Δt be some small positive number, and set $u^n \approx u(n\Delta t, \cdot)$, $u^{n+\frac{1}{2}} = (u^{n+1} + u^n)/2$. Given u^0, we define u^n to be the solution of

$$\left(u^{n+1}, v\right) + \Delta t \left(u^{n+\frac{1}{2}} u^{n+\frac{1}{2}}{}_x, v\right) + \Delta t \left(u^{n+\frac{1}{2}}{}_x, v_{xx}\right) = \left(u^n, v\right), \tag{3}$$

for all $v \in H^2(\mathbb{R})$ and $n \geq 0$. Assuming that this equation has a unique solution u^{n+1}, we can choose $v = u^{n+1} + u^n$ to get

$$\left\|u^{n+1}\right\|_{L^2(\mathbb{R})} = \left\|u^n\right\|_{L^2(\mathbb{R})} = \left\|u^0\right\|_{L^2(\mathbb{R})}. \tag{4}$$

Furthermore, by using a clever trick taken from Kato [10], we can get an á priori H^1 bound on u^n. Let R denote a positive constant, and introduce a smooth function φ satisfying;

- **a** $1 \leq \varphi(x) \leq 2R + 2$,
- **b** $\varphi'(x) = 1$ for $|x| < R$,
- **c** $\varphi'(x) = 0$ for $|x| \geq R + 1$
- **d** $0 \leq \varphi'(x) \leq 1$ for all x, and
- **e** $\left|\varphi^{(k)}(x)\right| \leq C\varphi(x)$ for all x and $k = 1, 2, 3$, and some constant C independent of R.

Assuming that u^n and u^{n+1} are in $H^2(\mathbb{R})$, $u^{n+\frac{1}{2}}\varphi$ is an admissible test function in (3), testing with this function yields

$$\begin{aligned} \frac{1}{2}\left\|u^{n+1}\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} &+ \Delta t \left(u^{n+\frac{1}{2}} u^{n+\frac{1}{2}}{}_x, u^{n+\frac{1}{2}}\varphi\right) \\ &+ \Delta t \left(u^{n+\frac{1}{2}}{}_x, \left(u^{n+\frac{1}{2}}\varphi\right)_{xx}\right) = \frac{1}{2}\left\|u^n\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{5}$$

To save space, we write $w = u^{n+\frac{1}{2}}$, then

$$\begin{aligned} \left(u^{n+\frac{1}{2}} u^{n+\frac{1}{2}}{}_x, u^{n+\frac{1}{2}}\varphi\right) &= -\frac{1}{2}\int_{\mathbb{R}} w^2 \left(w\varphi\right)_x \, dx \\ &= -\frac{1}{3}\int_{\mathbb{R}} w^3 \varphi_x \, dx. \end{aligned}$$